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$$1 + a + a^2 + a^3 + a^4 + \cdots = \frac{1}{1-a}$$

$$1 + a + a^2 + a^3 + \cdots + a^r = \frac{1-a^{r+1}}{1-a}$$

Last time we finished by looking at the matching worksheet of generating functions of sets of partitions. I want to move beyond "recognizing" when one generating function expression is a generating function for the number of partitions of a certain type to "deriving" the generating function expression for a set of partitions. Partitions because of the way that partitions are made up, they are sets of objects that are well suited for expressing the generating functions for the number of objects with algebraic expressions. This is not possible with most sets of combinatorial objects.

The study of partitions as combinatorial objects is often considered as part of the domain number theory since a partition n is a way of writing n as a sum of integers.

I gave the answers for the worksheet that I posted. I got very few questions about the answers but someone asked how to explain the generating function for the partitions of n with even parts and at most 4 parts of any given size. So I started to break down this set of partitions in two different ways.

Method 1: notice that if I let m_i be the number of parts of size i then every partition with even parts and at most 4 parts of any given size is a solution to the integer equation

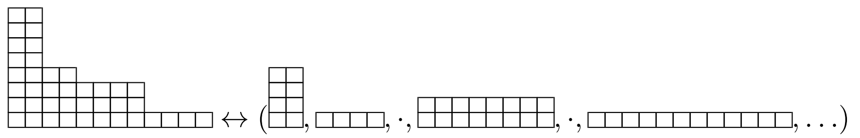
$$2m_2 + 4m_4 + 6m_6 + \cdots = n$$

where $0 \leq m_i \leq 4$. The generating function for this set of solutions is the product of the generating functions for the number of solutions to $2rm_{2r} = n$ with $0 \leq m_{2r} \leq 4$ for $r \geq 1$. We know that the generating function for the number of solutions to $2rm_{2r} = n$ with $0 \leq m_{2r} \leq 4$ is $1 + x^{2r} + x^{4r} + x^{6r} + x^{8r} = \frac{1-x^{10r}}{1-x^{2r}}$. Hence the generating function for the number of partitions with even parts and at most 4 parts of any given size is equal to $\prod_{r\geq 1} \frac{1-x^{10r}}{1-x^{2r}}$.

Method 2: I can break down this set of partitions into component pieces as a picture. Imagine that a partition partitions with even parts and at most 4 parts of any given size consists of at most 4 parts of size 2, at most 4 parts of size 4, at most 4 parts of size 6, etc.

In fact, a partition can be decomposed into a tuple consisting of parts of size $2r$ for $r \geq 1$ and there can be 0,1,2,3,or 4 parts of size $2r$.

For example the partition $(12, 8, 8, 4, 2, 2, 2, 2)$ can be decomposed into a tuple consisting of the parts $((2, 2, 2, 2), (4), (), (8, 8), (), (12), \ldots)$, or graphically



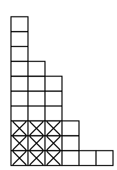
Again, I know that the generating function for the parts of size $2r$ is $1 + x^{2r} + x^{4r} + x^{6r} + x^{8r} = \frac{1-x^{10r}}{1-x^{2r}}$. Whenever we have a set of tuples like this we can apply the multiplication principle of generating functions and hence the generating function for the number of partitions of n with even parts and at most 4 parts of any given size is equal to $\prod_{r\geq 1} \frac{1-x^{10r}}{1-x^{2r}}$.

Using the same reasoning as in the above example and that we used in the last class, the generating function for the number of partitions of n with parts of size equal to k is $\mathcal{P}_{=k}(x) = \frac{1}{1-x^k}$. The generating function for the number of partitions of n with parts of size $\leq k$ is $\mathcal{P}_{\leq k}(x) = \prod_{i=1}^{k} \frac{1}{1-x^i}$.

We also said that the generating function for the partitions of n with no restriction is $\prod_{i\geq 1} \frac{1}{1-q^i}$ (this works by taking the limit of $\mathcal{P}_{\leq k}(x)$ as $k \to \infty$ and ensuring that for every coefficient of x^n that we might want to calculate is the same for $k > n$).

I also want to consider the partitions of length precisely equal to k, or alternatively if I take the transpose of these diagrams, this is the set of partitions whose first part is exactly equal to k. Every partition whose first part is exactly equal to k is isomorphic to a pair (X, Y) where X is some number ≥ 1 of parts of size equal to k and Y is a partition whose parts of size $\leq k-1$. The generating function for the partitions consisting of at least one part of size of size k is equal to $x^k + x^{2k} + x^{3k} + \cdots = \frac{x^k}{1-x^k}$. Therefore the generating function for the number of partitions whose first part is exactly equal to k is equal to $\frac{x^k}{1-x^k}\mathcal{P}_{\leq k-1}(x) = x^k \prod_{i=1}^{k} \frac{1}{1-x^i}$.

A Durfee square is the largest square that can fit in the diagram for a partition. For example, if my partition is $(6, 4, 4, 3, 3, 3, 2, 1, 1, 1)$ then the diagram of the partition is



and I can't put a larger square than 3×3 in that diagram. Now you can see from the example that I have drawn here that sitting on top of that Durfee square is a partition whose largest part is at most 3, and sitting off to the right of the Durfee square is a partition whose length is at most 3.

In general we can say that every partition that contains a $k \times k$ Durfee square is isomorphic to (a $k \times k$ Durfee square, a partition whose largest part is at most k, a partition whose length is at most k). By transposing a partition whose length is at most k, we have a partition whose largest part is at most k, therefore

g.f. for the number of partitions of n whose largest part is at most $k =$

g.f. for the number of partitions of n whose length is at most $k = \mathcal{P}_{\leq k}(x)$

By the MPofGFs, the generating function for partitions with a Durfee square equal to k is equal to

$$x^{k^2}\mathcal{P}_{\leq k}(x) = x^{k^2}\prod_{i=1}^{k}\frac{1}{(1-x^i)^2} .$$

Now if I also remark that every partition is either empty, or contains a Durfee square of size k for some $k \geq 1$, then I see that the generating function for all partitions (by the addition principle of generating functions) is equal to

$$= 1 + \sum_{k\geq 1} x^{k^2}\prod_{i=1}^{k}\frac{1}{(1-x^i)^2}$$

But we already knew that this was equal to an infinite product so we have shown the algebraic relation

$$\prod_{i\geq 1}\frac{1}{1-x^i} = 1 + \sum_{k\geq 1} x^{k^2}\prod_{i=1}^{k}\frac{1}{(1-x^i)^2} .$$

In case this is hard to comprehend, I will compute it on the computer and show you that the series are the same (at least for the first few terms.

```
sage: prod(1/(1-x^i) for i in range(1,10))
-1/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 - 1)*(x^6 - 1)*(x^7 - 1)*(x^8 - 1)*(x^9 - 1))
sage: taylor(prod(1/(1-x^i) for i in range(1,10)),x,0,10)
41*x^10 + 30*x^9 + 22*x^8 + 15*x^7 + 11*x^6 + 7*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
sage: taylor(1+x/(1-x)^2+x^4/((1-x)*(1-x^2))^2+x^9/((1-x)*(1-x^2)*(1-x^3))^2,x,0,10)
42*x^10 + 30*x^9 + 22*x^8 + 15*x^7 + 11*x^6 + 7*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
```

Notice that these two series differ in exactly the coefficient of x^{10}. This is because my first series is only the product of the terms $\frac{1}{1-x^i}$ for $1 \leq i < 10$ and so these two series will differ after the 10^{th} term.

I can also remark that every partition is empty or it has length equal to k for some $k \geq 1$. This implies that the generating function for the number of partitions of n is equal to (by my argument on p.2 of these notes,

$$1 + \sum_{k \geq 1} x^k \prod_{i=1}^{k} \frac{1}{1 - x^i}$$

This is a very powerful tool now that we have developed it properly, because we have shown that

$$\prod_{i \geq 1} \frac{1}{1 - x^i} = 1 + \sum_{k \geq 1} x^{k^2} \prod_{i=1}^{k} \frac{1}{(1 - x^i)^2} = 1 + \sum_{k \geq 1} x^k \prod_{i=1}^{k} \frac{1}{1 - x^i}$$

in other words, that an infinite product is equal to two different infinite sums just by arguing with pictures. Lets verify that this last sum is the same by calculating the example with the computer.

```
sage: f = 1+sum(x^i/prod(1-x^j for j in range(1,i+1)) for i in range(1,10))
sage: f
-x^9/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 - 1)*(x^6 - 1)*(x^7 -
1)*(x^8 - 1)*(x^9 - 1)) + x^8/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 -
1)*(x^6 - 1)*(x^7 - 1)*(x^8 - 1)) - x^7/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 -
1)*(x^5 - 1)*(x^6 - 1)*(x^7 - 1)) + x^6/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 -
1)*(x^5 - 1)*(x^6 - 1)) - x^5/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 -
1)) + x^4/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)) - x^3/((x - 1)*(x^2 -
1)*(x^3 - 1)) + x^2/((x - 1)*(x^2 - 1)) - x/(x - 1) + 1
sage: taylor(f,x,0,10)
41*x^10 + 30*x^9 + 22*x^8 + 15*x^7 + 11*x^6 + 7*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
```

Again, this series is wrong in the coefficient of x^{10} because I didn't add enough terms from my series, but I can easily change how many terms I add together and compute this series as high as I need.

I then gave you a worksheet where you were asked to do something similar to what I just did by giving an expression for the generating function for certain sets of partitions. I gave you each a problem from this and I really wanted everyone to go home and think about *one* problem. I then said that I would pick one at random that I would solve. I think that the next one that was available was (9) and when I looked at it I realized that the answer was complicated (I didn't know how to solve it). I made up these problems and some times it is possible to write down a sentence where the answer to that question is not 'nice.' This was one of those. I just had to change a few words and it corrected the problem to something that is solvable. The version that is on the website had the corrected version. Instead I solved number (10) in class and asked you to think about your question.

The instructions read: Apply the addition or the multiplication principle of generating functions to give the generating function for the following sequences of numbers.

(10) the number of partitions of n with with odd parts and a part will either occur 0 or an odd number of times

We decompose the partitions of n with odd parts that will occur 0 or an odd number of times into a tuple consisting of the parts of size 1, 3, 5, etc. Hence we can apply the MPofGFs to take the product for $i \geq 0$ of the parts of size $2i+1$ which occur 0 or an odd number of times. The generating function for those parts of size $2i+1$ which occur 0 or an odd number of times is equal to

$$1 + x^{2i+1} + x^{3(2i+1)} + x^{5(2i+1)} + x^{7(2i+1)} + \cdots =$$
$$1 + x^{2i+1}(1 + x^{2(2i+1)} + x^{4(2i+1)} + x^{6(2i+1)} + \cdots) =$$
$$1 + \frac{x^{2i+1}}{1 - x^{2(2i+1)}}$$

Therefore the generating function for the number of partitions of n with odd parts that will occur 0 or an odd number of times is equal to

$$\prod_{i \geq 0} \left(1 + \frac{x^{2i+1}}{1 - x^{2(2i+1)}} \right)$$