NOTES ON OCT 25, 2012

MIKE ZABROCKI

I wanted you do the problems on the worksheet that I gave you last time. Only a few people had done their problem. Even if it was a matter of just trying, it on the board so that we can see what was right and what was wrong, this was a good thing. We had a few people put up their answers:

(5) the number of partitions of n with at most 8 parts of any given size.
(28) the number of partitions of n with Durfee square of size 3×3 and all even parts.
(32) the number of partitions of n with a Durfee square of even size and all parts even.
hmmm...there was one more but it is 4 days later and I can't remember which one it was.

Someone asked me if I could post the answers and I agreed reluctantly that I would post the answers to some them. I am rescinding that statement. I will post the solutions/answers to any that people agree to present a solution to in class. I will check any answers that people want to verify with me through email. But if I post the answers, then this question becomes an entirely different problem. Rather than learning how to derive the answers yourself, you only have to match your answer/explanation against my expression. The matching worksheet already has a bunch of 'descriptions' and 'expressions' so if you need examples, then you have 18 of them right there. Here are three more right here.

The instructions read: Apply the addition or the multiplication principle of generating functions to give the generating function for the following sequences of numbers.

(5) the number of partitions of n with at most 8 parts of any given size.

The generating function for the partitions consisting only of parts of size i with at most 8 parts is equal to

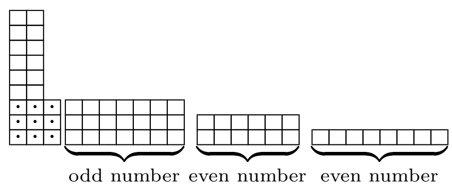
$$1 + x^i + x^{2i} + \cdots + x^{8i} = \frac{1 - x^{9i}}{1 - x^i}$$

The generating function for the number of partitions of n with at most 8 parts of any given size will be the product of the generating functions of the partitions consisting only of parts of size i with at most 8 parts for $i \geq 1$ because each partition can be decomposed into the parts of size i for $i \geq 1$. Therefore the generating function is equal to

$$\prod_{i \geq 0} \frac{1 - x^{9i}}{1 - x^i}$$

(28) the number of partitions of n with Durfee square of size 3×3 and all even parts.

A partition with a 3×3 Durfee square and all parts even consists of (a 3×3 Durfee square, a partition which lies above the Durfee square consisting only of parts of size 2, a partition that lies to the right of the Durfee square consisting of exactly three parts and all parts odd). The third entry in this tuple can be also be described as a partition consisting of an odd number of columns of size 3, an even number of columns of size 2 and an even number of columns of size 1.



This decomposition of a partition into these pieces implies that we can apply the MPofGFs and the the generating function for this whole set of partitions is equal to the product of the generating function for partitions with parts of size 2 only $= \frac{1}{1-x^2}$, the generating function for a 3×3 Durfee square $= x^3$, the generating function for an odd number of columns of size 3 $x^3 + x^9 + x^{15} + x^{21} + \cdots = \frac{x^3}{1-x^6}$, the generating function for the even number of columns of length $2 = 1 + x^4 + x^8 + x^{12} + \cdots = \frac{1}{1-x^4}$, the generating function for an even number of columns of length $1 = \frac{1}{1-x^2}$. Therefore the generating function for the number of partitions of n with Durfee square of size 3×3 and all even parts is equal to

$$\frac{1}{1-x^2} x^3 \frac{x^3}{(1-x^2)(1-x^4)(1-x^6)} = \frac{x^6}{(1-x^2)^2(1-x^4)(1-x^6)}$$

(32) the number of partitions of n with a Durfee square of even size and all parts even

A partition of n with a Durfee square of size $2k$ and all parts even consists of (a Durfee square of size $2k \times 2k$, a partition which lies above the Durfee square with all parts even and maximum part $2k$, a partition which lies to the right of the Durfee square where all parts are even and the length is less than or equal to $2k$). A "partition where all parts are even and the length is less than or equal to $2k$" can also be described as some even number of columns of size i for $1 \leq i \leq 2k$. Since the generating function for a even number of columns of size i is $\frac{1}{1-x^{2i}}$ hence the generating function for the partitions which lie to the right of the $2k \times 2k$ Durfee square is equal to $\prod_{i=1}^{2k} \frac{1}{1-x^{2i}}$. The partitions which are above the Durfee square consist only of even parts between 1 and $2k$, hence by the MPofGFs the generating function for the partitions which lies above the Durfee square with all parts even and maximum part $2k$ is equal to $\prod_{i=1}^{k} \frac{1}{1-x^{2i}}$. The Durfee square itself has generating function x^{4k^2}. Hence the generating function for partitions of n with a Durfee square of size $2k$ and all parts even is

$$x^{4k^2} \prod_{i=1}^{k} \frac{1}{1-x^{2i}} \prod_{i=1}^{2k} \frac{1}{1-x^{2i}} \ .$$

Now since all partitions of n with a Durfee square of even size and all parts even are either the empty partition or have a Durfee square of size $2k \times 2k$ for $k \geq 1$, then the generating function is

$$1 + \sum_{k \geq 1} x^{4k^2} \prod_{i=1}^{k} \frac{1}{1-x^{2i}} \prod_{i=1}^{2k} \frac{1}{1-x^{2i}} \ .$$

I expect you do the rest of these problems on your own. You won't learn any more by just reading. You have to learn to figure these out yourself.

The next thing that we are going to cover is Pólya enumeration. This requires that we know what the concept of a group is. If you have had a course in algebra before you have likely encountered the definition of a group before. You have all encountered the concept of a group. Let me tell you what one is and then show you that you have lots of examples:

A group is a set of elements G (possibly finite, possibly infinite) with a binary operation denoted $*$. That is $* : G \times G \to G$ and usually we denote it as $a * b \in G$ for $a, b \in G$. There are a few properties that this binary operation has in order to be a group.

(1) The product is associative, that is, for $a, b, c \in G$, $a * (b * c) = (a * b) * c$.
(2) There is an element $e \in G$ such that $g = e * g = g * e$ for all $g \in G$.
(3) For each element in $a \in G$, there is another element $\overline{a} \in G$ (called the inverse of a) such that $a * \overline{a} = \overline{a} * a = e$ (in many cases, we write the element $\overline{a} = a^{-1}$ but just remember that this does not mean $1/a$).

Here are some examples that you are probably familiar with:

(1) The integers $\mathbb{Z}$ with the binary operation of $+$. This example has the identity element 0 because $0 + a = a + 0 = a$ for all $a \in \mathbb{Z}$. For every integer a, $\overline{a} = -a$ has the property that $a + \overline{a} = 0$. Also addition is associative.
(2) The rational numbers except 0, $\mathbb{Q}\backslash\{0\}$, with multiplication $\cdot$ is the binary operation is an example of a group. In this example the identity element is 1 because $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{Q}\backslash\{0\}$. Moreover if $a \in \mathbb{Q}\backslash\{0\}$, then $\overline{a} = 1/a$ is an element such that $a\overline{a} = \overline{a}a = 1$. Also multiplication is associative.

(3) The group of permutations of 3, $G = \{123, 132, 213, 231, 312, 321\}$, with $a_1a_2a_3 \circ b_1b_2b_3 = b_{a_1}b_{a_2}b_{a_3}$. This example is a little different than the other examples because it is not immediately familiar to us that the multiplication is associative. In fact, it is since

$$(a_1a_2a_3 \circ b_1b_2b_3) \circ c_1c_2c_3 = b_{a_1}b_{a_2}b_{a_3} \circ c_1c_2c_3 = c_{b_{a_1}}c_{b_{a_2}}c_{b_{a_3}}$$
$$a_1a_2a_3 \circ (b_1b_2b_3 \circ c_1c_2c_3) = a_1a_2a_3 \circ c_{b_1}c_{b_2}c_{b_3}$$

and if you understand this properly, you can see that these are the same thing. Now the identity of this group is the element 123 since $123 \circ b_1b_2b_3 = b_1b_2b_3$. It is also the case that $a_1a_2a_3 \circ 123 = a_1a_2a_3$. You can check that the inverse element exists for for each of the 6 permutations in this group. Check that 123, 132, 213 and 321 are equal to their own inverse, 231 and 312 are inverses of each other.

OK these are three examples of groups and kind of cover a small range of examples, but groups are everywhere. In order to understand a definition clearly it is also a good idea to try to understand an example of something which is not a group.

(1) Take for example the integers except 0, $\mathbb{Z}\backslash\{0\}$, with the binary operation of $\cdot$ multiplication. This is an example of something which is not a group because there is nothing you can multiply the element 2 by in order to get 1 so there is no inverse of the element 2 (well, you can multiply it by $1/2$, but that isn't an integer and this is why $\mathbb{Q}\backslash\{0\}$ is a group and $\mathbb{Z}\backslash\{0\}$ is not).

(2) None of the integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ are groups with multiplication as the operation since they all include 0 and there is nothing you can multiply 0 by and get 1 (the identity element of the group). You might ask, what happens if I "throw in infinity and then define $0 \cdot \infty = \infty \cdot 0 = 1$" This is a great idea but it just kicks the problem to somewhere else in your group since $\infty \cdot (0 \cdot 2) = \infty \cdot 0 = 1$, but $(\infty \cdot 0) \cdot 2 = 1 \cdot 2 = 2$. It is the case that all of $\mathbb{Q}\backslash\{0\}$, $\mathbb{R}\backslash\{0\}$, $\mathbb{C}\backslash\{0\}$ are groups with multiplication as the binary operation, but if they include 0 then they are not a group.