NOTES ON NOV 6, 2012

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Last time we had some examples of groups:
Motions of a triangle with rotations only $\{e, R_{120}, R_{240}\}$
Motions of a triangle with rotations and flips $\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$
Motions of a square with rotations only $\{e, R_{90}, R_{180}, R_{270}\}$
Motions of a square with rotations and flips $\{e, R_{90}, R_{180}, R_{270}, F_H, F_V, F_{D_1} F_{D_2}\}$

Another good example of a group is the set $\{0, 1, 2, \ldots, n-1\}$ and the operation of addition *mod* n. If $n = 3$, the set of elements is $\{0, 1, 2\}$ and the operation is addition *mod* 3. The multiplication table looks like

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

I claim that we have seen this group before. Take a look at the following table from the last lecture:

$\circ$	e	R_{120}	R_{240}
e	e	R_{120}	R_{240}
R_{120}	R_{120}	R_{240}	e
R_{240}	R_{240}	e	R_{120}

It is the 'same' in some sense. What does it mean when I say that the groups are the same? I mean that there is a relabeling of the multiplication tables so that they are the same.

We say that a map f from a group $(G_1, *)$ to a group $(G_2, \cdot)$ is called a homomorphism if

$$f(g * h) = f(g) \cdot f(h) \tag{1}$$

for all g and h in G_1. If f is a bijection, then G_1 and G_2 are said to be isomorphic groups.

In the example above, we take $f(0) = e$, $f(1) = R_{120}$ and $f(2) = R_{240}$. Under this map, the tables look exactly the same and this is what is meant by equation (1).

Example 2:
the table of addition *mod* 4 looks like the following.

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

I claim that we also saw this table the other day when we had the group table

$\circ$	e	R_{90}	R_{180}	R_{270}
e	e	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	e
R_{180}	R_{180}	R_{270}	e	R_{90}
R_{270}	R_{270}	e	R_{90}	R_{180}

These tables are the 'same' because we can find a map $f(0) = e$ (the identity of each group always goes to the identity in a homomorphism), $f(1) = R_{90}$, $f(2) = R_{180}$, $f(3) = R_{270}$.

Example 3:
Consider the group $\{e, F_V\}$ with the multiplication table that looks like

$\circ$	e	F_V
e	e	F_V
F_V	F_V	e

Consider the map from $(\{e, F_V\}, \circ)$ to $(\{e, R_{90}, R_{180}, R_{270}\}, \circ)$ by the map $f(e) = e$ and $f(F_V) = R_{180}$. This map is a group homomorphism, because the group consisting of $(\{e, R_{180}\}, \circ)$ has the same multiplication table as $(\{e, F_V\}, \circ)$.

Example 4:
Consider the map from $(\{e, R_{90}, R_{180}, R_{270}\}, \circ)$ to $(\{e, F_V\}, \circ)$ such that $f(e) = f(R_{180}) = e$ and $f(R_{90}) = f(R_{270}) = F_V$. If we look at the image of the multiplication table in Example 2 and apply the map to it, we see

$\circ$	e	F_V	e	F_V
e	e	F_V	e	F_V
F_V	F_V	e	F_V	e
e	e	F_V	e	F_V
F_V	F_V	e	F_V	e

And this is all in agreement with the table from Example 3 (above).

Example 5:
We can also define a map $(\{0, 1, 2, 3\}, + \ mod\ 4)$ into itself that sends $f(i) = 2i \ mod\ 4$ for $i \in \{0, 1, 2, 3\}$. If you check, $f(i + j \ mod\ 4) = 2(i + j) \ mod\ 4$ and this is the same as $f(i) + f(j) = (2i \ mod\ 4) + (2j \ mod\ 4) \ mod\ 4 = 2(i + j) \ mod\ 4$.

I started babbling about how these functions are 1-1 and onto. I didn't want to spend too much class time defining these concepts, but they are important and come up everywhere in mathematics. When f maps G_1 to G_2 then G_1 is the domain and G_2 is called the co-domain. I like to use the language that an element $x \in G_1$ is 'sent to' an element $f(x)$ in G_2 so that I can say that intuitively 1-1 means that a function 'sends every element in the domain to a different element in the co-domain.' More precisely,

Definition 1. A function f that maps G_1 to G_2 is $1-1$ if $x, y \in G_1$ and $x \neq y$, then $f(x) \neq f(y)$.

Then I also like to say that an element y in the codomain is 'hit' if there is some x such that $f(x) = y$. A function is onto means that every element in the codomain is 'hit.' More precisely,

Definition 2. A function f that maps G_1 to G_2 is onto if for every $y \in G_2$, there is an element x in G_1 such that $f(x) = y$.

Example 3 is $1-1$, but not onto. Example 4 is onto, but not $1-1$. Example 5 is neither $1-1$ nor onto. Example 2 is both $1-1$ and onto (an isomorphism, bijection).

I then talked about the group of permutations of n and cycle notation. A permutation σ is a bijection from the numbers $\{1, 2, \ldots, n\}$ to the numbers $\{1, 2, \ldots, n\}$. We will represent σ in cycle notation, that is write it as

$$\sigma = (i_1, i_2, \ldots, i_{c_1})(j_1, j_2, \ldots, j_{c_2}) \cdots (\ell_1, \ell_2, \ldots, \ell_{c_r})$$

where the integers $\{1, 2, \ldots, n\}$ appear exactly once in the permutation. This notation means

$\sigma(i_k) = i_{k+1}$ for $1 \leq k < c_1$ and $\sigma(i_{c_1}) = i_1$
$\sigma(j_k) = j_{k+1}$ for $1 \leq k < c_2$ and $\sigma(j_{c_2}) = j_1$
$\vdots$
$\sigma(\ell_k) = \ell_{k+1}$ for $1 \leq k < c_r$ and $\sigma(\ell_{c_1}) = \ell_1$

The set if permutations of n represented this way with composition of permutations $\sigma \circ \tau$ is the permutation where $\sigma \circ \tau(i) = \sigma(\tau(i))$. I then tried to do an example with $n = 3$, but realized that the example is too small so I tried a larger example so that it is clear what I meant and how. Take $\sigma = (1, 3, 4)(2, 5, 6)(7)$ and $\tau = (1)(2, 3, 5)(4)(6, 7)$. The permutation σ should be read as "1 is sent to 3, 3 is sent to 4, 4 is sent to 1, 2 is sent to 5, 5 is sent to 6, 6 is sent to 2, 7 is sent to 7." or just $\sigma(1) = 3$, $\sigma(3) = 4$, $\sigma(4) = 1$, $\sigma(2) = 5$, $\sigma(5) = 6$, $\sigma(6) = 2$, $\sigma(7) = 7$. Similarly, the permutation τ should be read as $\tau(1) = 1$, $\tau(2) = 3$, $\tau(3) = 5$, $\tau(5) = 2$, $\tau(4) = 4$, $\tau(6) = 7$, $\tau(7) = 6$.

Let me try to indicate how we give the notation for $\sigma \circ \tau$, we start with by asking where 1 is sent (we can start with any integer, but this is a good place to start). In step 1, we have

$$\sigma \circ \tau = (1, \ldots$$

Then $\tau(1) = 1$ and $\sigma(\tau(1)) = \sigma(1) = 3$. Stated in words, τ sends 1 to 1 and σ sends it to 3. We record,

$$\sigma \circ \tau = (1, 3, \ldots$$

$\sigma(\tau(3)) = \sigma(5) = 6$ or in words 3 is sent to 5 under τ and 5 is sent to 6 under σ.

$$\sigma \circ \tau = (1, 3, 6, \ldots$$

$\sigma(\tau(6)) = \sigma(7) = 7$. In words again, 6 is sent to 7 under τ and 7 is sent to 7 under σ.

$$\sigma \circ \tau = (1, 3, 6, 7, \ldots$$

$\sigma(\tau(7)) = \sigma(6) = 2$. In other words, 7 is sent to 6 by τ and 6 is sent to 2 by σ.

$$\sigma \circ \tau = (1, 3, 6, 7, 2, \ldots$$

$\sigma(\tau(2)) = \sigma(3) = 4$.

$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4, \ldots$$

$\sigma(\tau(4)) = \sigma(4) = 1$.

$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4) \ldots$$

So far we have explained where everything except 5 is sent, so we add another cycle beginning with 5 (we would normally take any of the remaining elements that are not in a cycle yet).

$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4)(5, \ldots$$

$\sigma(\tau(5)) = \sigma(2) = 5$. That is τ sends 5 to 2 and σ sends 2 to 5. For this reason, we then close the parenthesis to indicate that 5 is sent to 5 under $\sigma \circ \tau$.

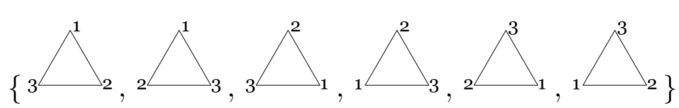
$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4)(5)$$

Since all of the integers 1 through 7 appear once in this expression we know that we are done.

Many of the examples we have considred above are not just groups, but the groups are motions of a square or a triangle. In other words they can be thought of as acting on a set of objects. We have a notion of this that I will introduce here.

Definition 3. A group action on a set X is a map $\bullet : G \times X \to X$ such that $e \bullet x = x$ for all $x \in X$ and $g \bullet (h \bullet x) = (gh) \bullet x$ for all $g, h \in G$ and $x \in X$.

For example, the set of motions of a triangle acts on the set



but you can also think of the motions as acting on just the vertices themselves $\{1, 2, 3\}$. For example $R_{120}(1) = 3$ (remember that we said that $R_{120} = (132)$). These groups are not really big enough to give a good clear example so I will wait until I have the group of the motions of a cube to give more examples.

Definition 4. The orbit of an element $x \in X$ is the set (it is a subset of X)

$$O_x = \{g \bullet x : g \in G\}$$

Definition 5. The stabilizer of an element $x \in X$ is a set (it is a subset of G)

$$Stab(x) = \{g \in G : g \bullet x = x\}$$

I will give some examples of the orbits and stabilizers when we have some better group actions. But for the moment consider the action of the group of motions of the square on the set of diagonals ⧅, ⧄ . Then $e \bullet$ ⧅ $= R_{180} \bullet$ ⧅ $= F_{D_1} \bullet$ ⧅ $= F_{D_2} \bullet$ ⧅ $=$ ⧅ while $R_{90} \bullet$ ⧅ $= R_{270} \bullet$ ⧅ $= F_V \bullet$ ⧅ $= F_H \bullet$ ⧅ $=$ ⧄ This defines a group action on the diagonals of the square (you will also need to figure out the action of the elements on ⧄, but these are enough to define the action.

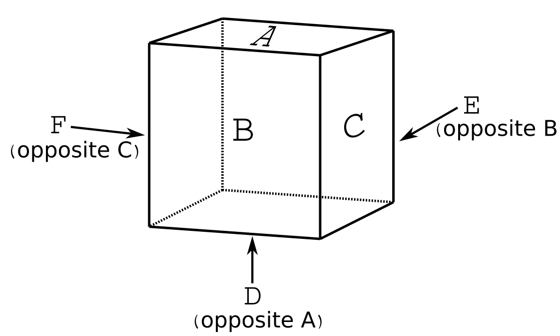
The orbit of ⧅ is $O_{⧅} = \{$ ⧅, ⧄ $\}$. The stabilizer of ⧅ is $Stab($⧅$) = \{e, R_{180}, F_V, F_H\}$.

Next I decided to say that we were ready to determine the group of motions of a cube. We can tell how many motions of a cube there are by a counting argument. If we label the 6 faces, then there are 6 ways of choosing which face will be up and then 4 ways of choosing which face will be in front. Therefore every motion of the cube is determined by these two steps so the number of motions of a cube is equal to $6 \cdot 4 = 24$.

Remark 6. A cube has 6 faces, 8 vertices and 12 edges. The number $24 = 4!$ which is equal to the number of permutations of $\{1, 2, 3, 4\}$. Here is a good question: is it possible to recognize the motions of the cube as the permutations of 4 things on the cube so that it is clear that these two groups are the same (isomorphic)?

So I gave you access to a cube to follow along because the cube I was working with was not big enough to see from a distance. It is a good idea in the following discussion to have a cube on hand to be able to better visualize what I am trying to communicate.

Take a cube and label the faces with the letters A, B, C, D, E, F.



One motion of the cube fixes all faces and is the identity of the group.

$$e = (A)(B)(C)(D)(E)(F)$$

Then it is possible to fix the face A and D and rotate the cube while keeping that face fixed. There are three rotations (besides the one where all faces are fixed.

$$(A)(D)(BCEF)$$
$$(A)(D)(BE)(CF)$$
$$(A)(D)(BFEC)$$

But we can also fix B and E and rotate around those faces

$$(B)(E)(ACDF)$$
$$(B)(E)(AD)(CF)$$
$$(B)(E)(AFDC)$$

and we can fix C and F and rotate around those faces

$$(C)(F)(ABDE)$$
$$(C)(F)(AD)(BE)$$
$$(C)(F)(AEDB)$$

Now we have found 10 motions of the cube and expressed them in terms of their action on the faces, but that is less than half since we are looking for 24. Now look at the top face labeled with A and pick one of the four edges that adjoins the faces B, C, E or F and then there is an edge which is furthest away from that edge. You can flip the cube across those two edges leaving them fixed and all the others edges are permuted. These correspond to the motions

$$(AB)(DE)(CF)$$
$$(AC)(DF)(BE)$$
$$(AE)(DB)(CF)$$
$$(AF)(DC)(BE)$$

There are two more of these kinds of flips where we flip across the edge which adjoins B and C and the corresponding edge between E and F and the edge adjoining B and F and E and C which are

$$(BC)(EF)(AD)$$
$$(BF)(EC)(AD)$$

Great, now we have 16 of the 24 motions of the cube. We need 24 in total. Exercise: find the other 8. Hint: look at the motions which fix diagonals across opposite corners. We haven't yet looked at those.