NOTES ON NOV 15, 2012

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In our last episode I showed you that,

$$|O_x| \cdot |Stab(x)| = |G| \ .$$

We are just one short calculation away from the result that we have been building up to for a while. Since I want to show it off, I am going to state it, give a bunch of examples (actually I will revisit some of the examples that we looked at already) and then I will justify why the formula is correct.

Theorem 1. *(Burnside's Lemma) Let G be a group which acts on a set of elements X,*

$$\text{The number of orbits when } G \text{ acts on } X = \frac{1}{|G|} \sum_{g \in G} \# \text{ of elements fixed by } g \ .$$

The reason that I say that we have now reached the point where we have given a formula for the examples that we have been discussing for the last couple of weeks is when we talk about colorings being equal we mean that they are in the same orbit. When we talk about different colorings, we are talking about two colorings being in different orbits under the action of G. So when we want to know how many different colorings there are, we want to know how many different orbits there are under the action of G and Burnside's Lemma is a formula for exactly that.

Remember on November 8 we figured out (by more or less writing down all possible colorings) the number of colorings of the vertices of a triangle under the action of three different groups, $\{e\}$, $\{e, R_{120}, R_{240}\}$ and $\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$. We arrived at the following table (there was a second column of this table but we will concentrate on just the first column. As an exercise figure out how the formula applies to the second column):

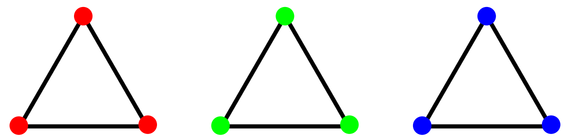
group	allowing repeated colors
$\{e\}$	3^3
$\{e, R_{120}, R_{240}\}$	11
$\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$	10

For the first row of this table it says that because the identity fixes all 3^3 possible colorings of the triangle that the number that are different under the group $\{e\}$ is equal to

$$\frac{1}{|\{e\}|} 3^3 = \frac{1}{1} \cdot 27 = 27 \ .$$

This example isn't very enlightening. But lets consider the other two.

When R_{120} and R_{240} act on the triangle, the only colorings that are fixed are those where all three vertices are colored exactly the same, that is:

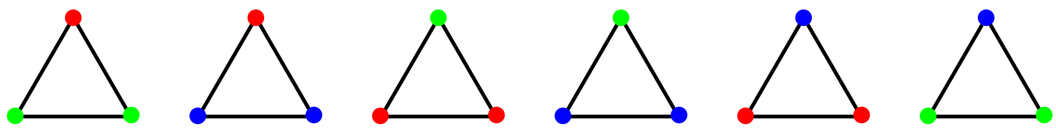


That means that the total number of different colorings under the group $\{e, R_{120}, R_{240}\}$ is equal to

$$\frac{1}{3}(3^3 + 3 + 3) = \frac{1}{3} \cdot 33 = 11$$

and this agrees with the table that we had calculated before.

If we look under the action of F_1, in addition to the three pictured colorings above, there are 6 others:



So in total, there are 9 colorings which are fixed by F_1. Similarly there are 9 which are fixed by F_2 and 9 which are fixed by F_3. Burnside's Lemma then tells us that the total number of different colorings by the action of this group is equal to

$$\frac{1}{6}(27 + 3 + 3 + 9 + 9 + 9) = \frac{1}{6} \cdot 60 = 10 \ .$$

Recall that the group elements have the following cycle structure

$$e = (1)(2)(3), R_{120} = (132), R_{240} = (123), F_1 = (1)(23), F_2 = (2)(13), F_3 = (3)(12) \ .$$

Unless there are other restrictions on the colors the number of elements in

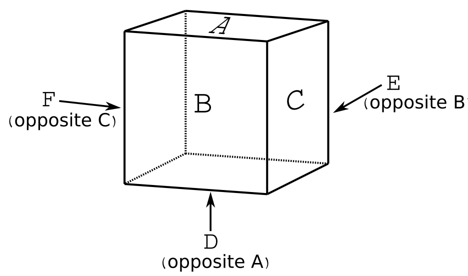
$$Fix(g) = (\#\text{ number of colors})^{(\#\text{ of cycles in } g)}$$

In particular we see $Fix(F_1) = Fix(F_2) = Fix(F_3) = 3^2$, $Fix(R_{120}) = Fix(R_{240}) = 3$ and $Fix(e) = 3^3$.

What is kind of cool about this formula is that just by looking at the expression, it is not clear that the order of the group in the denominator is going to cancel with the sum over the elements which are fixed by the group elements, but in the end it does. In fact, we can use this as a (weak) check that we haven't made any mistakes in our calculations by ensuring that the denominator does cancel with the numerator. If you get a rational number for the number of orbits, check again.

The reason this formula is useful, is that in general there are not that there are generally more colorings than there are group elements and another reason is that it is usually not that difficult to figure out how many elements are fixed by any particular group element g. Moreover, a lot of group elements have the same number of elements of x which are fixed by G.

Let me try to count the number of ways of coloring the faces of a cube with colors black and white such that two coloring are the same if one can be obtained from another by a motion of the cube. Fortunately we have already calculated the group of the motions of the cube. Label the faces of the cube with the letters A through F as in the following diagram.



Recall that the group of motions of the cube consisted of the following elements.

	$(C)(F)(ABDE)$	$(ABC)(DEF)$
$e = (A)(B)(C)(D)(E)(F)$	$(C)(F)(AD)(BE)$	$(ACB)(DFE)$
$(A)(D)(BCEF)$	$(C)(F)(AEDB)$	$(ABF)(DEC)$
$(A)(D)(BE)(CF)$	$(AB)(DE)(CF)$	$(AFB)(DCE)$
$(A)(D)(BFEC)$	$(AC)(DF)(BE)$	$(AEC)(DBF)$
$(B)(E)(ACDF)$	$(AE)(DB)(CF)$	$(ACE)(DFB)$
$(B)(E)(AD)(CF)$	$(AF)(DC)(BE)$	$(AEF)(DBC)$
$(B)(E)(AFDC)$	$(BC)(EF)(AD)$	$(AFE)(DCB)$
	$(BF)(EC)(AD)$	

- The identity $(A)(B)(C)(D)(E)(F)$ fixes all colorings and since we can choose b or w for each face, there are 2^6 colorings which are fixed by the identity.
- Say that we fix two faces then there are two types of permutation, those that rotate by ± 90 degrees (e.g. $(A)(D)(BCEF)$ or $(A)(D)(BFCE)$) and those that rotate by 180 degrees . The ones that rotate by $\pm 90°$ fix all colorings where all the 4 faces which move are the same color. There are two choices for the 4 faces and 2 choices for each of the two fixed faces. In total there are 2^3 colorings which are fixed by rotations by $\pm 90°$.
- The ones that rotate by $180°$ (e.g. $(A)(D)(BE)(CF)$) fix all colorings where the opposite faces that exchange are the same color. We have 2 choices for each of the two fixed faces and 2 choices for the two pairs of faces which exchange. We can read from the cycle structure of these permutations that there are 4 cycles and as long as each cycle has the same color and so in total there are 2^4 ways of coloring those faces.

- The permutations which fix an edge (e.g. $(AB)(DE)(CF)$) then there are three pairs of faces which are exchanged and they must be colored the same color and so there are 2^3 colorings which are fixed by these permutations.
- The permutations which fix a vertex and rotate by $\pm 120°$ (e.g. $(ABC)(DEF)$) must have the three faces which are all clustered around the vertex that is being rotated around all the same color therefore there are 2^2 colorings.

Look at the list of group elements above. We have:

- one identity element $(A)(B)(C)(D)(E)(F)$
- six rotations about two fixed faces by $\pm 90°$ (e.g. $(A)(D)(BCEF)$)
- three rotations about two fixed faces by $180°$ (e.g. $(A)(D)(BE)(CF)$)
- six flips about an edge (e.g. $(AB)(DE)(CF)$)
- eight rotations about a vertex by $\pm 120°$ (e.g. $(ABC)(DEF)$)

Burnside's Lemma then says that the number of colorings of a cube with black and white edges is equal to

$$\frac{1}{24}(2^6 + 6 \cdot 2^3 + 3 \cdot 2^4 + 6 \cdot 2^3 + 8 \cdot 2^2) = \frac{1}{24} \cdot 240 = 10 \ .$$

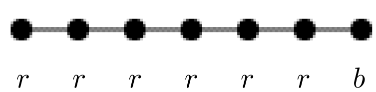
Now look back at your notes from October 30 and that was when we first started talking about colorings of the cube. I then said that the generating function for the number of colorings of the cube with black and white faces is:

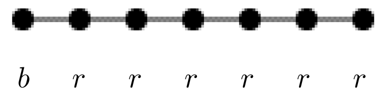
$$B^0W^6 + B^1W^5 + 2B^2W^4 + 2B^3W^3 + 2B^4W^2 + B^5W^1 + B^6W^0 \tag{1}$$

I will show you by the end of the class how we can give a formula for this generating function but if you add up all of the coefficients (the total number of colorings) it is $1 + 1 + 2 + 2 + 2 + 1 + 1 = 10$.

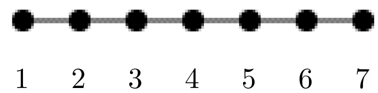
So I asked you on the homework to count the number of ways of coloring the vertices of the trees with 7 vertices using k colors such that two colorings are equal if one can be transformed to another by sending vertices to vertices and edges to edges. I thought I would show a single example of how I would like you to apply this formula to answer this question. Consider the colorings of the following graph.

Now notice that the group consisting of the identity and the motion which flips the tree backwards are the only two elements which preserve the tree structure. I want to count colorings where (for instance) the following two colorings are the same:





The way that we will go about doing this is to first label the vertices of the tree with the numbers 1 through 7 so that we can refer to them.



Then the two group elements which act on this tree are $e = (1)(2)(3)(4)(5)(6)(7)$ and $(17)(26)(35)(4)$. Now under the the identity element every coloring is fixed and there are k ways of coloring each of the 7 vertices so there are k^7 colorings fixed by e. Now a coloring which is fixed by $(17)(26)(35)(4)$ must have vertex 1 and vertex 7 colored the same, 2 and 6 must be colored the same, 3 and 5 must be colored the same and 4 can be colored independently. Since there are 4 different groups to color, in total $Fix((17)(26)(35)(4)) = k^4$ so Burnside's Lemma says that there are

$$\frac{1}{2}(k^7 + k^4)$$

different unique colorings of this graph. It is not clearly obvious that this result is even an integer for all values of k, but it can be checked both for k even and for k odd that the result is always an integer. If $k = 1$ we see for sure that the formula works because there is then exactly $1 = \frac{1}{2}(1 + 1)$ ways of coloring the graph with one color.

Great, now that we have three examples of how this formula works, I want to justify why it is true. Fortunately it is a short calculation from the orbit-stabilizer theorem.

I need to introduce one bit of shorthand notation. Define

$$Fix(g) = \#\{x : g \bullet x = g\}$$

so then Burnside's Lemma can then be restated as

The number of orbits when G acts on $X = \frac{1}{|G|} \sum_{g \in G} Fix(g)$.

In order to make the first part of my calculation clear I am going to make a table. Along the top of the table I label the columns by the x_i which are in the set $X = \{x_1, x_2, x_3, \ldots, x_{|X|}\}$. Along the left side of the table I label the rows by g_i which are the elements of $G = \{g_1, g_2, \ldots, g_{|G|}\}$ and in the body of the table I put a mark $\times$ in row g and column x in my table if x is fixed by g (that is, if $g \bullet x = x$).

So our table will typically look like the following where I am placing the $\times$ symbols in the table in a way to indicate that for the average group element, some elements are fixed and some are not. For the identity group element all elements are fixed (this is by the definition of group action).

$G\backslash X$	x_1	x_2	x_3	$\cdots$	$x_{\vert X\vert}$
$e = g_1$	$\times$	$\times$	$\times$	$\cdots$	$\times$
g_2		$\times$		$\cdots$	
g_3		$\times$	$\times$	$\cdots$	$\times$
g_4	$\times$		$\times$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$g_{\vert G\vert}$		$\times$		$\cdots$	$\times$

Now in the right hand column of the table I will count how many $\times$ symbols there are in each row. I have already given this quantity a name. The number of $\times$ symbols in the row indexed by g_i is $Fix(g_i)$, the number of elements of my set X which are fixed by g_i.

$G\backslash X$	x_1	x_2	x_3	$\cdots$	$x_{\vert X\vert}$	
$e = g_1$	$\times$	$\times$	$\times$	$\cdots$	$\times$	$Fix(g_1)$
g_2		$\times$		$\cdots$		$Fix(g_2)$
g_3		$\times$	$\times$	$\cdots$	$\times$	$Fix(g_3)$
g_4	$\times$		$\times$	$\cdots$		$Fix(g_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	
$g_{\vert G\vert}$		$\times$		$\cdots$	$\times$	$Fix(g_{\vert G\vert})$

Now below each column I will tally how many symbols $\times$ which appear in each column. This quantity has also been given a name. The number of $\times$ symbols which appear in the column indexed by x_i is the number of group elements which fix x_i or it is the number of elements in the stabilizer of x_i, $|Stab(x_i)|$

$G\backslash X$	x_1	x_2	x_3	$\cdots$	$x_{\vert X\vert}$	
$e = g_1$	$\times$	$\times$	$\times$	$\cdots$	$\times$	$Fix(g_1)$
g_2		$\times$		$\cdots$		$Fix(g_2)$
g_3		$\times$	$\times$	$\cdots$	$\times$	$Fix(g_3)$
g_4	$\times$		$\times$	$\cdots$		$Fix(g_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$g_{\vert G\vert}$		$\times$		$\cdots$	$\times$	$Fix(g_{\vert G\vert})$
	$\vert Stab(x_1)\vert$	$\vert Stab(x_2)\vert$	$\vert Stab(x_3)\vert$	$\cdots$	$\vert Stab(x_{\vert X\vert})\vert$	

So now if I sum the last row of this table it is equal to the total number of $\times$ symbols in the table and if I sum the last column it is also equal to the total number of $\times$ symbols in the table, hence we have that:

$$\sum_{x \in X} |Stab(x)| = \sum_{g \in G} Fix(g) \tag{2}$$

The right hand side of this equality is the right hand side of Burnside's Lemma multiplied by $|G|$. We also know from the orbit-stabilizer theorem that $|Stab(x)| = \frac{|G|}{|O_x|}$. Say that the set X breaks down into various orbits under the action of G and we number the orbits by a single representative:

$$X = O_{x_1} \uplus O_{x_2} \uplus O_{x_3} \uplus \cdots \uplus O_{x_{total\ \#\ orbits}}$$

Now then the left hand side of equation (2) is equal to

$$\begin{aligned}
\sum_{x \in X} |Stab(x)| &= \sum_{i=1}^{total\ \#\ orbits} \sum_{x \in O_{x_i}} |Stab(x)| \\
&= \sum_{i=1}^{total\ \#\ orbits} \sum_{x \in O_{x_i}} \frac{|G|}{|O_x|} \\
&= |G| \sum_{i=1}^{total\ \#\ orbits} \sum_{x \in O_{x_i}} \frac{1}{|O_{x_i}|} \\
&= |G| \sum_{i=1}^{total\ \#\ orbits} \frac{|O_{x_i}|}{|O_{x_i}|} \\
&= |G| \sum_{i=1}^{total\ \#\ orbits} 1 \\
&= |G| \cdot total\ \#\ orbits
\end{aligned}$$

Therefore we have show that $|G| \cdot total\ \#\ orbits = \sum_{g \in G} Fix(g)$, so

$$total\ \#\ orbits = \frac{1}{|G|} \sum_{g \in G} Fix(g)$$

Before I finished for the day I tried to squeeze in one more explanation. I wanted in fact to explain the example with the coloring with squares from the example above, and in particular I wanted to provide you with a formula for the generating function in equation (1).

Burnside's Lemma is quite robust because it just talks about a set X and it can be any set of colorings with a group action on them. The thing about group actions when they act on colorings is that the number of colors is independent of the element of the group acting on it so Burnside's Lemma says:

$$total\ \#\ orbits\ of\ colorings\ with\ a_i\ of\ i^{th}\ color\ appearing = \frac{1}{|G|} \sum_{g \in G} Fix_{with\ a_i\ color\ i}(g)$$

where $Fix_{with\ a_i\ color\ i}(g)$ represents the number of colorings with a_1 of color 1, a_2 of color 2, a_3 of color 3, etc. and the phrase *total # orbits of colorings with* a_i *of* i^{th} *color appearing* represents the subset of all of the distinct colorings with a_1 of color 1, a_2 of color 2, a_3 of color 3, etc. Because the group action does not affect the number of each color that appears, Burnside's Lemma applies.

Now sum over all weights $a = (a_1, a_2, a_3, \ldots)$ and multiply by $z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots$.

$$\sum_a (total\ \#\ orbits\ of\ colorings\ with\ a_i\ of\ i^{th}\ color\ appearing)\ z_1^{a_1}\ z_2^{a_2} z_3^{a_3} \cdots$$

$$= \sum_a \left(\frac{1}{|G|} \sum_{g \in G} Fix_{with\ a_i\ color\ i}(g) \right) z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_a (Fix_{with\ a_i\ color\ i}(g)\ z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots)$$

This is Polya's Theorem.

The left hand side of this equation is called the pattern inventory of the set. It is the generating function for the number of colorings where the coefficient of $z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots$ is the number of colorings with a_1 of color 1, a_2 of color 2, a_3 of color 3, etc.

The piece of the generating function $\sum_a (Fix_{with\ a_i\ color\ i}(g)\ z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots)$ on the right hand side is called the cycle index polynomial. If you look at it in one light Polya's Theorem *is* Burnside's Lemma with just a generating function replacing a number.

What is ingenious about this formula is that once we have the cycle structure of the group element g, the cycle index polynomial is usually very easy to compute because we can apply the multiplication principle of generating functions on the cycles. That is the generating function for the cycle index polynomial of g which is a product of cycles c_1, c_2, c_3, etc. is equal to the product of the cycle index polynomial for c_1 times the the cycle index polynomial for c_2 times the cycle index polynomial for c_3 times etc.

For instance, consider again the group of the cube and colorings with two colors B and W. Instead of the variables z_1 and z_2 I am going to use B and W in my cycle index polynomial to make it clearer which is the first color and the second color.

With the identity element $e = (A)(B)(C)(D)(E)(F)$, we have that

$$\sum_{i=0}^{6} (\#\text{colorings fixed by } e \text{ with } i \text{ W's } 6 - i \text{ B's})\ W^i B^{6-i} = (B + W)^6.$$

There are two ways of deducing this. The first is to say that the number of colorings with i white faces and $6 - i$ black faces is equal to $\binom{6}{i}$ and $\sum_{i=0}^{6} \binom{6}{i} W^i B^{6-i} = (B + W)^6$. The other way to deduce it is to say that it is equal to the product of the generating function for the colorings of the face A times the generating function for the colorings of the face B times $\cdots$ the generating function for the number of colorings of the face $F = (B + W)^6$.

Consider the element $(A)(D)(BCEF)$. The generating function for the colorings which are fixed by this element is the product of the generating function for the colorings of the face A times the generating function for the colorings of the face D times the generating function for the colorings of B, C, E and F. These last 4 need to be done together because they all need to be the same to color. Therefore

$$\sum_a (\#\text{colorings fixed by } (A)(D)(BCEF) \text{ with } a_1 \text{ W's } a_2 \text{ B's})\, W^{a_1} B^{a_2} = (B+W)^2(B^4+W^4).$$

Unless there are extra conditions placed on the colorings, it is easy to write down the generating function for the colorings of a group element with cycles of length $r_1, r_2, \ldots, r_\ell$ because each cycle will have the same color so the generating function for a cycle of size r_1 is always $B^{r_1} + W^{r_1}$ and the generating function for the colorings which are fixed by g is

$$(B^{r_1} + W^{r_1})(B^{r_2} + W^{r_2}) \cdots (B^{r_\ell} + W^{r_\ell}) \ .$$

Therefore the rest of group elements have cycle index polynomials

- $(B+W)^6$ for one identity element $(A)(B)(C)(D)(E)(F)$
- $(B+W)^2(B^4+W^4)$ six rotations about two fixed faces by $\pm 90°$ (e.g. $(A)(D)(BCEF)$)
- $(B+W)^2(B^2+W^2)^2$ three rotations about two fixed faces by $180°$ (e.g. $(A)(D)(BE)(CF)$)
- $(B^2+W^2)^3$ six flips about an edge (e.g. $(AB)(DE)(CF)$)
- $(B^3+W^3)^2$ eight rotations about a vertex by $\pm 120°$ (e.g. $(ABC)(DEF)$)

Therefore, the generating function for the colorings with black and white faces is given by the expression

$$\frac{1}{24}((B+W)^6+6(B+W)^2(B^4+W^4)+3(B+W)^2(B^2+W^2)^2+6(B^2+W^2)^3+8(B^3+W^3)^2)$$

I asked Sage to expand this result for me and I find that it gives exactly the result in equation (1).

```
sage: B,W = var('B','W')
sage: expand(1/24*((B+W)^6 + 6*(B+W)^2*(B^4+W^4) + 3*(B+W)^2*(B^2+W^2)^2 + \
6*(B^2+W^2)^3 + 8*(B^3+W^3)^2))
B^6 + B^5*W + 2*B^4*W^2 + 2*B^3*W^3 + 2*B^2*W^4 + B*W^5 + W^6
```

I can more or less count the number of colorings of the cube with two colors by hand, but increasing the number of colors or the size of the object does not significantly increase the complexity of using this formula but it does make counting these colorings by hand significantly more complicated. Consider colorings of the cube with three colors (just as an example).

```
sage: R,G,B = var('R,G,B')
sage: expand(1/24*((R+G+B)^6 + 6*(R+G+B)^2*(R^4+G^4+B^4) \
+ 3*(R+G+B)^2*(R^2+G^2+B^2)^2 + 6*(R^2+G^2+B^2)^3 + 8*(R^3+G^3+B^3)^2))
B^6 + B^5*G + B^5*R + 2*B^4*G^2 + 2*B^4*G*R + 2*B^4*R^2 + 2*B^3*G^3
+ 3*B^3*G^2*R + 3*B^3*G*R^2 + 2*B^3*R^3 + 2*B^2*G^4 + 3*B^2*G^3*R + 6*B^2*G^2*R^2
+ 3*B^2*G*R^3 + 2*B^2*R^4 + B*G^5 + 2*B*G^4*R + 3*B*G^3*R^2 + 3*B*G^2*R^3
+ 2*B*G*R^4 + B*R^5 + G^6 + G^5*R + 2*G^4*R^2 + 2*G^3*R^3 + 2*G^2*R^4 + G*R^5 + R^6
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