NOTES ON SEPT 13-18, 2012

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Last time I gave a name to

- $S(n,k)$:= number of set partitions of $[n]$ into k parts. This only makes sense for $n \geq 1$ and $1 \leq k \leq n$. For other values we need to choose a convention that makes sense. We also have that $S(n,1) = S(n,n) = 1$ and $S(n,k) = S(n-1,k-1)+kS(n-1,k)$ for $1 < k < n$.

This time I started to add to the sets we can count and give recursive formulas for:

- $P(n)$:= the number of permutations of n, and by default $P(0) = P(1) = 1$ (I said $n!$, but I thought about it and $n!$ in most places is referring to the algebraic definition, these are the same thing, but its better to reserve the symbol $n!$ for the algebraic definition).
- $\binom{n}{k}$:= the number of ways of choosing k elements from the set $\{1, 2, \ldots, n\}$
- $B(n)$:= the number of set partitions of $\{1, 2, \ldots, n\}$ (and $B(0) = 1$)
- $s'(n,k)$:= the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k cycles (and $s(n,k) = (-1)^{n-k}s'(n,k)$)

$S(n,k)$ are called the Stirling numbers of the second kind

$s(n,k)$ are called the Stirling numbers of the first kind and $s'(n,k)$ are called the (unsigned) Stirling numbers of the first kind.

Remark, that they are related to algebra by the formula (try some examples of these formulas to make sure you are comfortable with them, you are asked to prove the last one as a homework exercise exactly as I had done it in class for the the first one.

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k$$

$$(x)_n = \sum_{k=1}^{n} s(n,k)x^k$$

$$x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}$$

$$(x)^{(n)} = \sum_{k=1}^{n} s'(n,k)x^n$$

Remark: These things are discussed in section 1.4, 1.8, 2.9 and 2.13 of the book, but it would be good to familiarize yourself with the other sections in chapter 1 and 2.

I explained bijectively why for $n > 1$,

$$P(n) = nP(n-1)\ . \tag{1}$$

Conclusion $P(n) = n! = n(n-1)(n-2)\cdots 2\cdot 1$.

I explained bijectively that for $n \geq 1$ and $0 \leq k \leq n$,

$$n! = \binom{n}{k} k!(n-k)! \tag{2}$$

Conclusion $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

I explained that while it was a simple application of the addition principle to state why

$$B(n) = \sum_{k=1}^{n} S(n,k) \tag{3}$$

that we could also argue bijectively that

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(n-k-1) \tag{4}$$

This allowed us to compute $B(n)$ for $1 \leq n \leq 4$ in two different ways:

Table of $S(n,k)$

$n\backslash k$	1	2	3	4	5	6	$B(n)$
1	1						1
2	1	1					2
3	1	3	1				5
4	1	7	6	1			15
5	1	15	25	10	1		52
6	1	31	90	65	15	1	203

also

$$B(2) = \binom{1}{0}B(1) + \binom{1}{1}B(0) = 2$$

$$B(3) = \binom{2}{0}B(2) + \binom{2}{1}B(1) + \binom{2}{2}B(0) = 2+2+1 = 5$$

$$B(4) = \binom{3}{0}B(3) + \binom{3}{1}B(2) + \binom{3}{2}B(1) + \binom{3}{3}B(0) = 5 + 3\cdot 2 + 3\cdot 1 + 1 = 15$$

Notice that this second way is not particularly more efficient than calculating with the Stirling numbers, but it does proved a very different sort of formula.

We also computed the first few examples of the Stirling numbers of the first kind (with signs). I used the expansion $(x)_n = \sum_{k=1}^{n} s(n,k)x^k$ (which you are supposed to prove for homework).

$$x(x-1) = x^2 - x$$

$$x(x-1)(x-2) = x^3 - 3x^2 + 2x$$

$$x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$$

Table of $s(n,k)$

$n\backslash k$	1	2	3	4	5	6	$n!$
1	1						1
2	-1	1					2
3	2	-3	1				6
4	-6	11	-6	1			24

We also computed a few of these values by the use of the definition $s(n,k)$.

$s'(4,3) = 6$: this is because the permutations of 1234 into 3 cycles have two fixed points and two elements swapped. Once we know which are the fixed points are then the permutation is determined. So the number of permutations of 1234 into 3 cycles is $\binom{4}{2} = 6 =$ the number of ways of picking two fixed points.

I computed also that $s'(4,2) = 11$ because there are two choices, either there is one fixed point and the remaining three elements are in a cycle or there are two pairs of elements that are being exchanged. In the first case, there are 4 possible fixed points and two possible cycles, so there are 8 permutations with one fixed point and two cycles. In the second case we can choose to swap 1 with 2,3 or 4 and the remaining two of those choices will also be swapped, so there are 3 permutations with two 2-cycles. In total there are 8+3=11 permutations with two cycles.

Stirling numbers and their relationship with polynomials is discussed in section 2.9 and 2.13.

I wanted to give some examples of counting sets of objects like those that were in the homework. One of the best examples of this is counting poker hands. Poker is a card game played with a 52 card deck with 13 values for the cards and 4 suits. Poker hands are ranked by how common a hand is. For instance, there are 13×48 possible 4-of-a-kind hands because we can choose which value appears 4 times in a 4-of-a-kind hand plus one extra card from the remaining 48 cards in the deck. There are also 40 straight flush hands because there are 4 possible suits and 10 possible straights. Therefore a straight flush beats a 4-of-a-kind.

The number of 5 card poker

If we want to count a set of possible hands we need to apply the multiplication principle and the addition principle sometimes in creative ways.

For instance if I want to count the number of hands that have exactly one pair, then I note that every pair is determined by the following 4 peices of information.

- the value of the card that appears twice
- the values of the other three cards (all different and not the same as the last value)
- the two suits used by the pair
- a suit used by the smallest of the three cards
- a suit used by the middle of the three cards
- a suit used by the largest of the three cards

That is I am saying that if I am given a particular five card hand with containing exactly a pair, then the 6 pieces of information are all that is necessary to determine the hand and the hand determines the information. Therefore the set of hands containing a pair are in bijection with tuples containing the information in that list. For example the hand $3\heartsuit$, $5\diamondsuit$, $7\diamondsuit$, $7\clubsuit$, $10\spadesuit$ and this is isomorphic to this list $(7, \{3, 5, 10\}, \{\diamondsuit, \clubsuit\}, \heartsuit, \diamondsuit, \spadesuit)$.

Now there are 13 ways of choosing the card that appears twice; $\binom{12}{3}$ ways of choosing a set of three elements from the 12 values that are not the pair; there are $\binom{4}{2}$ possible sets for the suits which appear in the pair; there are 4 suits possible for the non-pair card; 4 suits for the second non-pair card; 4 suits for the third non-pair card. In total there are

$$13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4 \cdot 4 \cdot 4 =$$

is the number of hands with exactly one pair.

I also counted the number of hands with exactly two pairs. The following information completely determines a hand that has a two pair.

- two values (an upper and a lower) which will each appear twice in the hand
- two suits of the 4 for the lower value
- two suits of the 4 for the upper value
- a last card which is any of the $52 - 8$ cards which don't have a value of the pair.

Again, I can frame this in terms of a bijection with a list of information. A hand with 5 cards in it is in bijection with a list containing 4 pieces of information. For instance the hand $3\heartsuit$, $3\clubsuit$, $7\clubsuit$, $K\spadesuit$, $K\clubsuit$ is a hand with two pairs. It is in bijection with $(\{3, K\}, \{\heartsuit, \clubsuit\}, \{\spadesuit, \clubsuit\}, 7\clubsuit)$.

Now the number of possible lists are easy to count by the multiplication principle. There are $\binom{13}{2}$ choices for the values of the pairs. There are $\binom{4}{2}$ possible sets of two suits from the set $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$ and there are 44 remaining cards. Therefore the number of hands with two pairs is

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 \ .$$

I mentioned that you should avoid subtraction if you can, but there are always exceptions to that rule. The reason I would like you to avoid subtraction is that it is hard to explain clearly. One obvious exception to that rule is the number of straight hands which are not straight flushes. By basic counting techniques we know that there are $10 \cdot (4^5 - 4)$ possible straight hands which don't have a flush because there are 10 possible straights (that begin with A through 10 as the lowest card) and there are 4^5 ways of picking a suit for each of the cards of the straight, BUT we have to subtract off the number of ways that all suits are the same. This is the easy way of explaining how to pick the suits and it involves subtraction.

There is the hard way of explaining how to pick the suits too that only involves addition. Let me count the value $4^5 - 4$ in a different way. We know that either the hand contains 2, 3 or 4 different suits.

- Say there are two different suits, a first suit and a second suit. There are $S(5, 2)$ ways of distributing the suits among the 5 cards (for example the set partition $\{\{1, 3\}, \{2, 4, 5\}\}$ means that the straight will have the form $1X$, $2Y$ $3X$, $4Y$, $5Y$ where X is the first suit and Y is the second suit and $1, 2, 3, 4, 5$ will be the values in the straight) and 4 ways of picking the first suit and 3 ways of picking the second suit. There are $S(5, 2) \cdot 4 \cdot 3$ ways of having two suits in your poker hand which is a straight.
- Say that there are 3 different suits that appear in the hand. There are $S(5, 3)$ ways of distributing the suits among the 5 cards, 4 ways of picking the first suit, 3 ways of picking the second suit (can't be the same as the last), 2 choices for the third suit. Thus there are $S(5, 3) \cdot 4 \cdot 3 \cdot 2$ ways of having a straight hand with exactly 3 different suits.
- Say that all 4 suits appear in your hand. Then there are $S(5, 4)$ ways of distributing the suits among the cards in the hand and 4! ways of assigning an order to the suits.

Therefore the number of ways of choosing suits for a hand such that not all suits are the same is $S(5, 2)\cdot 4\cdot 3+S(5, 3)\cdot 4\cdot 3\cdot 2+S(5, 4)\cdot 4\cdot 3\cdot 2\cdot 1 = 15\cdot 4\cdot 3+25\cdot 4\cdot 3\cdot 2+10\cdot 4\cdot 3\cdot 2\cdot 1 = 4^5-4$. This should be compared with the identity that we have already proven

$$4^5 = S(5, 1) \cdot 4 + S(5, 2) \cdot 4 \cdot 3 + S(5, 3) \cdot 4 \cdot 3 \cdot 2 + S(5, 4) \cdot 4 \cdot 3 \cdot 2 \cdot 1 + S(5, 5) \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0$$

This is clearly NOT better than explaining $4^5 - 4$, just different. No matter how you explain why your answer is what it is, you should always get the same value in the end. Computing a value in two different ways gives you a way of checking your answer.

The types of poker hands are:

- royal flush - $10, J, Q, K, A$ all the same suit
- straight flush (sometimes these first two sets are combined): a sequence of 5 cards in order all with the same suit
- 4 of a kind
- flush - five cards all one suit not a straight
- full house - a pair and a three of a kind
- straight - five cards whose values are in a 5 card sequence and it is not the case they all have the same suit
- 3 of a kind
- two pairs
- pair
- none of the above

A really good exercise is to figure out a way of counting the number of each of these sets using only addition (like I said, this is sometimes the more complicated way of coming up with the answer) and then add them all up and check that they add up to $\binom{52}{5}$ (the number of ways of picking 5 cards from a deck of 52).

Counting hands of cards is discussed in section 1.13 in the book.

I also discussed a little about proving combinatorial identities. I don't want an algebraic proof. In the case of the identities that are on the homework, two of the three of them would be completely trivial to show with a little algebra (what I mean is an algebraic proof is just show $n^3 = n(n-1)(n-2) + 3n(n-1) + n$ and you are done with the first one). I talked about in particular

$$n^3 = (n)_1 + 3(n)_2 + (n)_3$$

You are to find a set such that the number of elements in the set is n^3. Here are some ideas (but there are an infinite number of possible answers).

- The number of sequences of three values where there are n choices for each of the three values.
- The number of three digit numbers base n (I then proceeded to count in binary (base 2) on my fingers, and then lost track at 20 and I was still on my right hand).
- The number of ways of painting three different rooms with n different colors

After a while these combinatorial interpretations all begin to sound the same. Then you need to find a combinatorial interpretation for each of the terms on the right hand side and find a bijection to them. For instance $(n)_3 = n(n-1)(n-2)$ is equal to "the number of ways of painting three different rooms with n different colors where all the colors used are different."

Combinatorial identities are discussed in section 2.3 in the book.