NOTES ON SEPT 27, 2012

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We started to experiment a bit with generating functions and manipulate them and come up with formulas. We wrote down a bunch of sequences that we were able to give formulas for their generating functions. Recall that on Tuesday I had said look at the sequences

$$\binom{0}{k}, \binom{1}{k}, \binom{2}{k}, \binom{3}{k}, \binom{4}{k}, \ldots$$

If you look for $k = 1, 2, 3, \ldots$ then you can conjecture that there is a relatively simple formula for the generating function

$$\binom{0}{k} + \binom{1}{k} x \binom{2}{k} x^2 \binom{3}{k} x^3 \binom{4}{k} x^4 + \cdots = \sum_{n \geq 0} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} .$$

Proof. Take the derivative of $1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n\geq 0} x^n = \frac{1}{1-x}$. We have (by a quick induction argument), that

$$\frac{d^k}{dx^k} \frac{1}{1-x} = \frac{k!}{(1-x)^{k+1}}$$

We also know that

$$\frac{d^k}{dx^k} \frac{1}{1-x} = \frac{d^k}{dx^k} \sum_{n\geq 0} x^n = \sum_{n\geq 0} n(n-1)(n-2)\cdots(n-k+1)x^{n-k}$$

Therefore

$$\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} x^{n-k}$$

But the binomial coefficient $\binom{n}{k}$ is exactly the coefficient in this sum since

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2\cdot 1}{k!(n-k)(n-k-1)\cdots 2\cdot 1} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} .$$

Therefore

$$\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0} \binom{n}{k} x^{n-k}$$

and

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n\geq 0} \binom{n}{k} x^n . \quad \square$$

This corresponds to looking at columns of Pascal's triangle. We can also look at rows

$$\binom{1}{0}, \binom{1}{1}, \binom{1}{2}, \binom{1}{3}, \binom{1}{4}, \ldots = 1, 1, 0, 0, 0, 0, \ldots$$
$$\binom{2}{0}, \binom{2}{1}, \binom{2}{2}, \binom{2}{3}, \binom{2}{4}, \ldots = 1, 2, 1, 0, 0, 0, \ldots$$
$$\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}, \binom{3}{4}, \ldots = 1, 3, 3, 1, 0, 0, \ldots$$
$$\vdots$$
$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots$$

These have generating functions $(1+x)$, $(1+x)^2$, $(1+x)^3$ and the general sequence has generating function

$$(1+x)^n = \sum_{k \geq 0} \binom{n}{k} x^k \ .$$

Proof. left to the reader. easiest to do this by induction on n. □

Then I suggested we look at sequences like $1, 2, 3, 4, 5, \ldots$ and $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$ and $1^3, 2^3, 3^3, 4^3, 5^5, \ldots$. I looked at

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots = \sum_{n \geq 0}(n+1)x^n.$$

If you multiply by x and then take the derivative then you get the generating function for the squares because

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + \cdots = \sum_{n \geq 0}(n+1)x^{n+1}$$

and

$$\frac{1+x}{(1-x)^3} = \frac{d}{dx}\frac{x}{(1-x)^2} = 1 + 4x + 9x^2 + 16x^3 + 25x^4 + 36x^5 + \cdots = \sum_{n \geq 0}(n+1)^2 x^n \ .$$

At this point I was using the computer at a regular basis. I went to the website `www.sagemath.org` and I had registered for an account. I used that account to do some of the calculations.

```
sage: taylor((1+x)/(1-x)^3,x,0,15)

256*x^15 + 225*x^14 + 196*x^13 + 169*x^12 + 144*x^11 + 121*x^10 + 100*x^9
+ 81*x^8 + 64*x^7 + 49*x^6 + 36*x^5 + 25*x^4 + 16*x^3 + 9*x^2 + 4*x + 1

sage: diff(x/(1-x)^2,x)
```

```
1/(x - 1)^2 - 2*x/(x - 1)^3

sage: factor(diff(x/(1-x)^2,x))

-(x + 1)/(x - 1)^3
```

The first command takes the taylor series of the expression $\frac{1+x}{(1-x)^3}$, the second command takes the derivative of $\frac{x}{(1-x)^2}$ and (since that wasn't presented as a single fraction) the third command factored the rational expression and showed it was equal to $-\frac{x+1}{(x-1)^3}$.

Then I said, what if I wanted to come up with a formula for the generating function $\sum_{n\geq 0}(n+1)^3x^n$? I should just multiply the last result by x and then differentiate. We find that

$$\frac{d}{dx}\left(x\frac{1+x}{(1-x)^3}\right) = \frac{d}{dx}\left(x\sum_{n\geq 0}(n+1)^2x^n\right) = \sum_{n\geq 0}(n+1)^3x^n$$

and I can use the computer to determine that:

```
sage: factor(diff(x*(1+x)/(1-x)^3,x))

(x^2 + 4*x + 1)/(x - 1)^4

sage: taylor((1+4*x+x^2)/(1-x)^4,x,0,14)

3375*x^14 + 2744*x^13 + 2197*x^12 + 1728*x^11 + 1331*x^10 + 1000*x^9 + 729*x^8
+ 512*x^7 + 343*x^6 + 216*x^5 + 125*x^4 + 64*x^3 + 27*x^2 + 8*x + 1
```

The last thing that I decided to do was look at what to do if we have a generating function for a sequence $a_0, a_1, a_2, a_3, \ldots$

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

and I want to know what the generating function was for the sequence of just the even terms $a_0, a_2, a_4, a_6, \ldots$. If I set $x \to -x$ then I see that

$$f(-x) = a_0 + a_1(-x) + a_2(-x)^2 + a_3(-x)^3 + a_4(-x)^4 + a_5(-x)^5 + a_6(-x)^6 + \cdots$$

then notice if we add $f(x) + f(-x)$ we have

$$f(x) + f(-x) = 2a_0 + 2a_2x^2 + 2a_4x^4 + 2a_6x^6 + \cdots$$

and then divide by 2

$$\frac{1}{2}(f(x) + f(-x)) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots$$

and then replace $x \to \sqrt{x}$, so that

$$\frac{1}{2}(f(\sqrt{x}) + f(-\sqrt{x})) = a_0 + a_2x + a_4x^2 + a_6x^3 + \cdots$$

and this is the generating function for the sequence $a_0, a_2, a_4, a_6, \ldots$.

I then did an example on the computer to convince us that it works as it should.

```
sage: f = (1+4*x+x^2)/(1-x)^4

sage: (f.subs(x=sqrt(x))+f.subs(x=-sqrt(x)))/2

1/2*(x + 4*sqrt(x) + 1)/(sqrt(x) - 1)^4 + 1/2*(x - 4*sqrt(x) + 1)/(sqrt(x) + 1)^4

sage: factor(_)

(x + 1)*(x^2 + 22*x + 1)/((sqrt(x) - 1)^4*(sqrt(x) + 1)^4)
```

(*) when I wrote `factor(_)` sage acted with the function `factor` on the last result the _ refers to the last result.

This says that the generating function for the odd cubes is given by

$$\sum_{n\geq 0}(2n+1)^3x^n = \frac{(1+x)(1+22x+x^2)}{(1-x)^4}$$

(note that if I was patient enough to do all the algebra on the blackboard I could have derived the same result by hand, but I don't have time to do all that in class).

If I want to check my answer, I find that

```
sage: taylor((1+x)*(1+22*x+x^2)/(1-x)^4,x,0,10)

9261*x^10 + 6859*x^9 + 4913*x^8 + 3375*x^7 + 2197*x^6 + 1331*x^5 + 729*x^4 +
343*x^3 + 125*x^2 + 27*x + 1
```

I suggested that for next time that you try to do the same thing except pick out every third term. What you need to do this is a little complex numbers. Everyone told me that this isn't common knowledge (as I assumed it should be). So here is a little summary:

$$i = \sqrt{-1}$$
$$e^{\theta i} = cos(\theta) + isin(\theta)$$

an r^{th} root of unity is given by the formula $\zeta_r = e^{2\pi i/r}$ because

$$(\zeta_r)^r = e^{2\pi i} = cos(2\pi) + isin(2\pi) = 1$$
$$1 + \zeta_r + \zeta_r^2 + \cdots + \zeta_r^{r-1} = 1\ .$$

What you want to do to generalize the formula for picking out every other term to every third term is to think of -1 as a second root of unity since $\zeta_2 = e^{\pi i} = -1$ and $1 + \zeta_2 = 0$ so instead of $f(x) + f(\zeta_2 x)$ you want something else.