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I started off with an example that used complex numbers and this was not quite familiar to everyone. Last time we figured out that if we started with the generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, then it is possible to give the generating function for just the even terms in a three step process. First add $A(x)$ and $A(-x)$ and we find the generating function for $2a_0, 0, 2a_2, 0, 2a_4, 0, 2a_6, 0, \ldots$

$$A(x) + A(-x) = 2a_0 + 2a_2 x^2 + 2a_4 x^4 + 2a_6 x^6 + \cdots$$

then divide by two and find the generating function for $a_0, 0, a_2, 0, a_4, 0, a_6, 0, \ldots$,

$$\frac{1}{2}(A(x) + A(-x)) = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots$$

then replace x with $\sqrt{x}$ and find

$$\frac{1}{2}(A(\sqrt{x}) + A(-\sqrt{x})) = a_0 + a_2 x + a_4 x^2 + a_6 x^3 + \cdots$$

and this is the generating function for the sequence $a_0, a_2, a_4, a_6, \ldots$.

Now what if we wanted to generalize this process to pick out every third term instead of every second? For this we need to know why every other term of the sequence cancelled. The reason is that $1^r + (-1)^r = 0$ if r is odd, and $1^r + (-1)^r = 2$ if r is even. The generalization of this statement is in complex numbers.

$$e^{ix} = cos(x) + isin(x)$$

If I set $\zeta_r = e^{2\pi i/r}$ (this is a definition), then $(\zeta_r)^r = e^{2\pi i} = 1$ and so

$$0 = (\zeta_r)^r - 1 = (\zeta_r - 1)(\zeta_r^{r-1} + \zeta_r^{r-2} + \cdots + \zeta_r + 1)$$

now since $\zeta_r - 1$ is not 0 and the product is 0, this means that $\zeta_r^{r-1} + \zeta_r^{r-2} + \cdots + \zeta_r + 1 = 0$.

Example: $\zeta_3 = e^{2\pi i/3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\zeta_3^2 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2})^2 = \frac{1}{4} - \frac{3}{4}) - i\frac{\sqrt{3}}{2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Then we see

$$\zeta_3 + \zeta_3^2 + 1 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2}) + (-\frac{1}{2} - i\frac{\sqrt{3}}{2}) + 1 = 0\ .$$

Example: $\zeta_2 = -1$, and $\zeta_2 + 1 = 0$.

Example: $\zeta_4 = I$, and $\zeta_4^2 = -1$, $\zeta_4^3 = -I$ and so

$$1 + \zeta_4 + \zeta_4^2 + \zeta_4^3 = 1 + I - 1 - I = 0 \ .$$

This is what we use to generalize what we did for the $r = 2$ case to pick out every other term. Step 1 is to add up $A(x)$, $A(\zeta_3 x)$ and $A(\zeta_3^2 x)$. We see

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$A(\zeta_3 x) = a_0 + a_1 \zeta_3 x + a_2 \zeta_3^2 x^2 + a_3 x^3 + a_4 \zeta_3 x^4 + a_5 \zeta_3^2 x^5 + a_6 x^6 + \cdots$$

$$A(\zeta_3^2 x) = a_0 + a_1 \zeta_3^2 x + a_2 \zeta_3 x^2 + a_3 x^3 + a_4 \zeta_3^2 x^4 + a_5 \zeta_3 x^5 + a_6 x^6 + \cdots$$

and so their sum is equal to

$$A(x)+A(\zeta_3 x)+A(\zeta_3^2 x) = 3a_0 + a_1(1+\zeta_3+\zeta_3^2)x + a_2(1+\zeta_3^2+\zeta_3)x^2 + 3a_3x^3 + a_4(1+\zeta_3+\zeta_3^2)x^4$$

$$+ a_5(1 + \zeta_3^2 + \zeta_3)x^5 + 3a_6 x^6 + \cdots = 3a_0 + 3a_3 x^3 + 3a_6 x^6 + \cdots$$

This is the generating function for $3a_0, 0, 0, 3a_3, 0, 0, 3a_6, 0, 0, \ldots$. The next step is to divide this expression by 3 and the final step is to replace x by $\sqrt[3]{x}$. The final result is

$$\frac{1}{3}(A(\sqrt[3]{x}) + A(\zeta_3 \sqrt[3]{x}) + A(\zeta_3^2 \sqrt[3]{x})) = a_0 + a_3 x + a_6 x^2 + \cdots$$

The example that I did in class worked OK on the computer, but I didn't know how to make the computer do the algebra for us. The suggestion was that we take every third term of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$. If we do this we should get the same expression back. We find that

```
sage: taylor(1/(1-x),x,0,10)
x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: zeta3 = exp(2*pi*I/3); zeta3
sage: taylor(1/(1-x) + 1/(1-zeta3*x) + 1/(1-zeta3^2*x),x,0,10)
3*x^9 + 3*x^6 + 3*x^3 + 3
sage: taylor(1/3*(1/(1-x) + 1/(1-zeta3*x) + 1/(1-zeta3^2*x)).subs(x=x^(1/3)),x,0,10)
x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
```

So what this shows is that the series for this expression has the same series as `1/(1-x)` but I couldn't figure out how to make the package do the simplification and show that

$$\frac{1}{3}\left(\frac{1}{1-\sqrt[3]{x}} + \frac{1}{1-\zeta_3\sqrt[3]{x}} + \frac{1}{1-\zeta_3^2\sqrt[3]{x}}\right) = \frac{1}{1-x}$$

instead you have to do the algebra yourself....

$$\frac{1}{3}\left(\frac{1}{1-\sqrt[3]{x}}+\frac{1}{1-\zeta_3\sqrt[3]{x}}+\frac{1}{1-\zeta_3^2\sqrt[3]{x}}\right)=$$

$$\frac{1}{3}\left(\frac{(1-\zeta_3\sqrt[3]{x})(1-\zeta_3^2\sqrt[3]{x})+(1-\sqrt[3]{x})(1-\zeta_3^2\sqrt[3]{x})+(1-\sqrt[3]{x})(1-\zeta_3\sqrt[3]{x})}{(1-\sqrt[3]{x})(1-\zeta_3\sqrt[3]{x})(1-\zeta_3^2\sqrt[3]{x})}\right)=$$

$$\frac{1}{3}\left(\frac{(1-\zeta_3\sqrt[3]{x}-\zeta_3^2\sqrt[3]{x}+x^{2/3})+(1-\sqrt[3]{x}-\zeta_3^2\sqrt[3]{x}+\zeta_3^2x^{2/3})+(1-\sqrt[3]{x}-\zeta_3\sqrt[3]{x}+\zeta_3x^{2/3})}{(1-\sqrt[3]{x}-\zeta_3\sqrt[3]{x}+\zeta_3x^{2/3})(1-\zeta_3^2\sqrt[3]{x})}\right)=$$

$$\frac{1}{3}\left(\frac{3}{(1-\sqrt[3]{x}-\zeta_3\sqrt[3]{x}+\zeta_3x^{2/3}-\zeta_3^2\sqrt[3]{x}+\zeta_3^2x^{2/3}+x^{2/3}-x)}\right)=$$

$$\frac{1}{3}\left(\frac{3}{(1-x)}\right)=$$

$$\frac{1}{(1-x)}=$$

I recommend that you experiment both by hand and with the computer to see that complex numbers work the way that you think that they do. Since $x^2-y^2=(x+y)(x-y)$ then it is also the case that $x^2+y^2=(x+iy)(x-iy)$. So it is possible to divide one complex number of the form $a+bi$ by $c+di$ (where a,b,c,d are all real numbers) and you will be able to put it in the form $e+fi$ by multiplying by the appropriate thing to clear the denominator of the complex numbers. So as an exercise, I suggest you try to show that

$$\frac{a+bi}{c+di}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}\ .$$

I then jumped to simpler example. How do we shift the generating function for a sequence and multiply by coefficients, etc.

sequence	generating function	expression
$a_0, a_1, a_2, a_3, a_4, a_5, a_6, \ldots$	$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$	$A(x)$
$0, 0, 0, a_0, a_1, a_2, a_3, a_4, \ldots$	$a_0x^3 + a_1x^4 + a_2x^5 + a_3x^6 + a_4x^7 + \cdots$	$x^3A(x)$
$a_3, a_4, a_5, a_6, a_7, a_8, a_9, \ldots$	$a_3 + a_4x + a_5x^2 + a_6x^3 + a_7x^4 + a_8x^5 + \cdots$	$(A(x) - a_0 - a_1x - a_2x^2)/x^3$
$0a_0, 1a_1, 2a_2, 3a_3, 4a_4, \ldots$	$a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + 5a_5x^5 + \cdots$	$xA'(x)$
$\binom{0}{k}a_0, \binom{1}{k}a_1, \binom{2}{k}a_2, \binom{3}{k}a_3, \ldots$	$\sum_{n\geq 0}\binom{n}{k}a_n$	$x^kA^{(k)}(x)$

I then showed how to get the generating function for the Fibonacci numbers. Define $F_0 = 1$ and $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. The first few terms of the sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

By definition the generating function is given by

$$F(x) = \sum_{n\geq 0} F_n x^n = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots .$$

It follows then that,

$$\begin{aligned} F(x) &= 1 + x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n \\ &= 1 + (x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots) + (F_0 x^2 + F_1 x^3 + F_2 x^4 + F_3 x^5 + \cdots) \\ &= 1 + xF(x) + x^2 F(x) \end{aligned}$$

By rearranging the terms of this formula we have

$$F(x) - xF(x) - x^2 F(x) = (1 - x - x^2) F(x) = 1$$

so

$$F(x) = \frac{1}{1 - x - x^2} .$$

I quickly checked this on sage and found

```
sage: taylor(1/(1-x-x^2),x,0,10)
89*x^10 + 55*x^9 + 34*x^8 + 21*x^7 + 13*x^6 + 8*x^5 + 5*x^4 + 3*x^3
+ 2*x^2 + x + 1
```

I will use this next time to show formulas that relate the Fibonacci numbers.

Exercises: Find formulas for the following generating functions (you don't need to simplify the expressions, but use the tools that we have developed in the last few days to write down an an expression).

(1) $\sum_{n\geq 0} F_{3n} x^n$
(2) $\sum_{k\geq 0} \binom{n}{2k} x^{2k}$
(3) $\sum_{n\geq 0} \binom{2n+1}{3} x^n$
(4) $\sum_{n\geq 0} \binom{n}{3} x^{2n+1}$
(5) $\sum_{n\geq 0} \binom{n}{2} F_n x^n$
(6) $\sum_{n\geq 0} \binom{n}{2} F_{n+4} x^n$
(7) $\sum_{n\geq 0} \binom{n+2}{2} \binom{n-2}{2} x^n$

Given that $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$ are the generating functions for the sequences $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$ respectively, find an expression for the generating function for the following sequences.

(8) $a_0, 2a_1, 4a_2, 8a_3, 16a_4, \ldots$
(9) $0, a_1, 2^2 a_2, 3^2 a_3, 4^2 a_4, 5^2 a_5, \ldots$
(10) $a_0, a_0, a_1, a_1, a_2, a_2, a_3, a_3, \ldots$
(11) $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$
(12) $a_0, b_1, a_2, b_3, a_4, b_5, \ldots$

(13) $a_1, a_5, a_9, a_{13}, a_{15}, a_{19}, \ldots$
(14) $a_0 + b_0, a_0 - b_0, a_1 + b_1, a_1 - b_1, a_2 + b_2, a_2 - b_2, \ldots$