NOTES ON OCT 4, 2012

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I had planned a midterm on Oct 11. I can't be there that day. I am canceling my office hours that day and I will be available on Tuesday Oct 9 from 4-5pm instead. I am tempted to give a take home miterm instead of the in class one (which is very limited by the time). We will see....

Consider what happens when you multiply two generating functions

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

and

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \cdots$$

then if you expand it term by term you see

$$f(x)g(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots$$

Observe that in the expansion that the coefficient of x^n (for $n = 0, 1, 2$ only because I didn't go further) that the subscripts of a_ib_j add up to the exponent of x. If we expand all terms of the series then we reason that this always happens and we have that the coefficient of x^n is $\sum_{i+j=n} a_ib_j$. That is,

$$f(x)g(x) = \sum_{n\geq 0} \left(\sum_{i+j=n} a_ib_j \right) x^n \ .$$

I declared that if a_r and b_s have a combinatorial meaning, then a_rb_s has a combinatorial meaning and so does $\sum_{r+s=n} a_rb_s$. I formulated this as a mathematical principle.

Principle 1. (**The Multiplication Principle of Generating Functions**) *Assume that a_r is equal to the number of widgets of 'size' r and b_s is equal to the number of doodles of 'size' s, then we say that $f(x)$ is the generating function for the number of widgets of 'size' n and $g(x)$ is the generating function for the number of doodles of 'size' n and*

$$f(x)g(x) = \sum_{n\geq 0} \left(\sum_{i+j=n} a_ib_j \right) x^n$$

is the generating function for the pairs of elements (x, y) where x is a widget of 'size' i and y is a doodle of 'size' j with $i + j = n$.

So what I have done is I have applied the addition principle and the multiplication principle to count such pairs (x, y) where x is a widget and y is a doodle where I break the set of pairs of 'size' n into those where x is of size i and y is of size $n - i$. In order to make the statement of the principle above I had to apply the addition principle so that the widget was of size i where $0 \leq i \leq n$.

Remark 2. *I intentionally put the word 'size' in quotes because I haven't been super precise about what I mean. This really means that if I group the objects that I am calling widgets into groups by a grading (something that happens often in combinatorics) then the word 'size' here represents an association with the grading. The word 'size' may not be accurate. Consider the example below when I am talking about change for n cents, then the 'size' in that case means the number of cents. I am using 'size' in an abstract way to mean whatever term you are grading by.*

Remark 3. *This notation of expressing $f(x)$ as the generating function for the number of widgets of 'size' n and $g(x)$ the generating function for the number of doodles of 'size' n is my own. You won't see it in the textbook and if you google the words 'widgets' and 'doodles' you are likely to find web pages written by me. I just find this a convenient way to think about combinatorics of generating functions in the case when the generating functions are for sequences of non-negative integers and there is a combinatorial interpretation for these integers. If $f(x)$ is the g.f. for widgets and $g(x)$ is the g.f. for doodles then $f(x)g(x)$ is this generating function for pairs consisting of a widget and a doodle (i.e. a widget-doodle).*

Let me give you an example of something we can apply this principle to. Consider the number of non-negative solutions to the equation $x_1 + x_2 = n$ for $n \geq 0$. If I write the generating function for the number of such solutions I can compute it in two different ways and get the same answer.

The first way is I will just look and notice that the non-negative solutions to the equation $x_1 + x_2 = n$ are $(x_1, x_2) \in \{(n, 0), (n-1, 1), (n-2, 2), \ldots, (0, n)\}$. Therefore the number of solutions to $x_1 + x_2 = n$ is equal to $n+1$ and the generating function $\sum_{n\geq 0}(n+1)x^n = \frac{1}{(1-x)^2}$.

Now let me try to compute the same thing using the multiplication principle of generating functions (MPofGFs). The a solution to $x_1 + x_2 = n$ is isomorphic to a solution to a pair (x_1, x_2) whose sum is n. By MPofGFs we have that

$$\sum_{n\geq 0}(\#\text{pairs } (x_1, x_2) \text{ s.t. } x_1 + x_2 = n)x^n = \left(\sum_{n\geq 0}(\#\text{solutions to the equation } x_1 = n)x^n\right)^2$$

But the number of solutions to the equation $x_1 = n$ is equal to 1 for all $n \geq 0$ so

$$\sum_{n\geq 0}(\#\text{solutions to the equation } x_1 = n)x^n = \frac{1}{1-x}$$

and hence

$$\sum_{n\geq 0}(\#\text{pairs } (x_1, x_2) \text{ s.t. } x_1 + x_2 = n)x^n = \frac{1}{(1-x)^2}\ .$$

I know it seems a kind of trivial example, but we have shown that the number of solutions to $x_1 + x_2 = n$ has generating function equal to $1/(1-x)^2$ in two different ways. Lets try to expand this.

The generating function for the number of non-negative solutions to

$$x_1 + x_2 + x_3 + x_4 = n$$

is equal to the number of tuples (x_1, x_2, x_3, x_4) where $x_1 + x_2 + x_3 + x_4 = n$ which is equal to the number of pairs (X, Y) where X is a pair (x_1, x_2) with $x_1 + x_2 = i$ and Y is a pair (x_3, x_4) with $x_3 + x_4 = n - i$. By the MPofGFs we know that

$$\sum_{n\geq 0}(\#\text{pairs } (X,Y) \text{ s.t. } X \text{ is a solution to } x_1+x_2=i \text{ and } Y \text{ is a solution to } x_3+x_4=n-i)x^n$$

$$= \left(\sum_{n\geq 0}(\text{pairs } (x_1,x_2) \text{ s.t. } x_1+x_2=n)x^n\right)^2 = \left(\frac{1}{(1-x)^2}\right)^2 = \frac{1}{(1-x)^4}\ .$$

In general, we can apply the MPofGFs multiple times to show that

$$\sum_{n\geq 0}\#(\text{number of solutions to } x_1+x_2+\cdots+x_k=n)x^n = \frac{1}{(1-x)^k}.$$

The thing is that this is something that we have already discussed in this class

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0}\binom{n+k-1}{k-1}x^n$$

so the number of solutions to $x_1 + x_2 + \cdots + x_k = n$ is equal to $\binom{n+k-1}{k-1}$. We had discussed this before that the number of solutions is equal to the number of sequences of n dots $\bullet$ and $k-1$ bars $|$.

Here is an example of a problem that we can apply these ideas to: "How many ways are there of making change for 78 using pennies, nickels, dimes, and quarters." The answer is equivalent to the number of tuples (p, n, d, q) such that $p+5n+10d+25q = n$. If we apply MPofGFs, then this is the product of the generating functions for solutions to $p = N$, the solutions to $5n = N$, the solutions to $10d = N$, the solutions to $25q = N$ and these sequences have respective generating functions $\frac{1}{1-x}$, $\frac{1}{1-x^5}$, $\frac{1}{1-x^{10}}$ and $\frac{1}{1-x^{25}}$.

Therefore the generating function for the number of ways of making change for N cents with pennies, nickels, dimes and quarters is

$$C(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}\ .$$

If in particular I wanted the number of ways of making change for 78 cents I would go to the computer and ask:

```
sage: taylor(1/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)),x,0,78).coefficient(x^78)
121
sage: taylor(1/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)),x,0,10)
```

```
4*x^10 + 2*x^9 + 2*x^8 + 2*x^7 + 2*x^6 + 2*x^5 + x^4 + x^3 + x^2 + x + 1
```

I also calculated here the ways of making change for N cents for $0 \leq N \leq 10$ and I notice that the number ways of making change for 10 cents is $4 = \#\{$10 pennies; 1 nickel, 5 pennies; 2 nickels; one dime $\}$ and this agrees with the answer that the generating function returns.

I then wanted to demonstrate that you can throw in some pretty crazy conditions on your combinatorial problem and calculating the number of such solutions is still a matter of breaking up the problem into pieces where you can either add or multiply generating functions. As long as your combinatorial condition has a nice expression for the generating function, then applying this tool works really well.

So, for instance say that in addition that you wanted to make change for N cents where you also have an American quarter and two American nickels but as many Canadian pennies, nickels, dimes and quarters as you want. You can break the combinatorial problem into the number of tuples (X, Y, Z) where X is some way of taking change for I cents with Canadian coins, Y is some way of taking change for J cents using the American quarter or not, Z is some way of making change for K cents using the two American nickels. We want to know how may ways there are of making change for N cents, so we will take the coefficient of x^N in the expression for the product of generating functions.

We already know that the generating function for the first part of the tuple is $C(x)$ (given above). With the American quarter we can make change either for 0 cents or 25 cents and only in one way each so the generating function is $1 + x^{25}$. With the two American nickels we can make change for $0, 5$ or 10 cents only and there is exactly one way of doing that (the nickels are indistinguishable), then the generating function is equal to $1 + x^5 + x^{10}$. Therefore the generating function for N cents where you also have an American quarter and two American nickels but as many Canadian pennies, nickels, dimes and quarters as you want is equal to

$$C(x)(1 + x^{25})(1 + x^5 + x^{10}) = \frac{(1 + x^{25})(1 + x^5 + x^{10})}{(1 - x)(1 - x^5)(1 - x^{10})(1 - x^{25})} .$$

We can compute the number of these by asking the computer:

```
sage: taylor((1+x^5+x^10)*(1+x^25)/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)),
       x,0,78).coefficient(x^78)
430
```

We can also use generating function to derive combinatorial identites. Recall that last time, I showed that the generating function for the Fibonacci numbers is $1/(1 - x - x^2) = F(x) = \sum_{n \geq 0} F_n x^n$. Then we can rewrite this as

$$F(x) = \frac{1}{1 - (x + x^2)} = \sum_{n \geq 0} (x + x^2)^n = \sum_{n \geq 0} (1 + x)^n x^n$$

We also know that $(1+x)^n$ is the generating function for the binomial coefficients $(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k$ therefore

$$F(x) = \sum_{n\geq 0} \sum_{k\geq 0} \binom{n}{k} x^{n+k}$$

If I take the coefficient of x^m in both sides of this equation I find that

$$F_m = \sum_{n+k=m} \binom{n}{k} .$$

For example

$$F_5 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} + \binom{2}{3} + \binom{1}{4} + \binom{0}{5} .$$

I know that $\binom{2}{3}+\binom{1}{4}+\binom{0}{5} = 0$ and $\binom{5}{0} = 1$, $\binom{4}{1} = 4$ and $\binom{3}{2} = 3$, therefore $F_5 = 1+4+3 = 8$.

$$F_6 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1+5+6+1 = 13$$

and this agrees with our generating function

```
sage: taylor(1/(1-x-x^2),x,0,8)
34*x^8 + 21*x^7 + 13*x^6 + 8*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
```

We can also derive a second equation for the Fibonacci numbers. If you apply the quadratic formula to $1 - x - x^2 = 0$ you obtain that $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{1-\sqrt{5}}{2}$ are the roots of the equation. Check explicitly that $\phi\overline{\phi} = -1$ and $\phi + \overline{\phi} = 1$, therefore

$$(1-\phi x)(1-\overline{\phi}x) = 1 - \phi x - \overline{\phi}x + \phi\overline{\phi}x^2 = 1 - x - x^2$$

Now if I have a rational function of the form $\frac{1}{(1-\phi x)(1-\overline{\phi}x)}$ then there is this technique that you probably learned in calculus that says that there exists A and B such that

$$F(x) = \frac{1}{(1-\phi x)(1-\overline{\phi}x)} = \frac{A}{(1-\phi x)} + \frac{B}{(1-\overline{\phi}x)} .$$

If we take the coefficient of x^m in both sides of this equation we find that

$$F_m = A\phi^m + B\overline{\phi}^m .$$

If you solve for A and B by saying that since $A(1-\overline{\phi}x) + B(1-\phi x) = 1$, then let $x = 1/\overline{\phi}$ to see that $B = \frac{1}{1-\phi/\overline{\phi}} = \frac{\overline{\phi}}{\overline{\phi}-\phi} = -\frac{\overline{\phi}}{\sqrt{5}}$ and let $x = 1/\phi$ so then $A = \frac{\phi}{\phi-\overline{\phi}} = \frac{\phi}{\sqrt{5}}$. We conclude

$$F_m = \frac{\phi^{m+1}}{\sqrt{5}} - \frac{\overline{\phi}^{m+1}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{m+1} .$$

which (at least to me) is kind of hard to believe until you do this by hand or test it out on the computer.

```
sage: expand( 1/sqrt(5)*(((1+sqrt(5))/2)^6-((1-sqrt(5))/2)^6))
8
sage: expand( 1/sqrt(5)*(((1+sqrt(5))/2)^7-((1-sqrt(5))/2)^7))
13
```

Exercises: I was asked about what would be questions at the level of a test question. You should be able to answer

(a) during a timed exam,
(b) during a timed exam (this should be at a slightly harder level maybe you will need a computer to get the numerical value)
(c) would be at the level of a homework problem or a take home exam
(d) should be considered a challenge and (while doable) may take a while to complete

On the following two questions find a generating function representing the sequence for a all n. Take the coefficient of x^n for the n specified in the problem

(1) How many ways are there making change for $n = \$1.00$ with pennies, nickels, dimes and quarters such that:
  (a) there are an even number of nickels and no pennies ?
  (b) such that there at most 6 nickels ?
  (c) the total number of nickels and dimes is even ?
  (d) the total number of pennies, dimes and quarters is even ?
(2) How many ways are there of placing $n = 50$ balls in 10 distinguished boxes such that:
  (a) there is no restriction ?
  (b) there are at most 17 balls in the first box ?
  (c) the first 4 boxes have at most 10 of the balls ?
  (d) the first 4 boxes have at least half of the balls ?

(3) (a) Find the generating function for the sequence $a_0, 2a_1, a_2, 2a_3, a_4, 2a_5, a_6, 2a_7, \ldots$ in terms of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.
  (b) Find the generating function for the sequence $a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6, \ldots$ in terms of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.
  (c) On the homework assignment you were to arrive at an expresion for $L(x) = \sum_{n\geq 0} L_n x^n = (1 + 2x)/(1 - x - x^2)$. Using the formula for the product of generating functions, what is the coefficient of x^n in the generating function $\frac{1}{1+2x} L(x)$? Conclude a formula relating the Fibonacci numbers and the Lucas numbers because $F(x) = \frac{1}{1+2x} L(x)$.
  (d) Given $D_0 = 1$, $D_1 = a$ and $D_{n+1} = aD_n + bD_{n-1}$ where a, b are unknowns. The entry sequence D_n will be a polynomial in a and b. Find the coefficient of $a^r b^s$.

Enumeration problems (this is not related to generating functions, it is a review of the types of combinatorial problems that came up on the last homework): Make sure that you explain your answer as completely as possible. It is not sufficient to give just a numerical answer, you must give an explanation why your answer is correct.

(4) How many 5 card hands ...
   (a) contain a three of a kind and a 3 values in a row ?
   (b) contain a three of a kind sequence and 3 values in a row that are not all of the same suit ?
   (c) contain a three of a kind sequence and 3 values in a row that are not all of the same suit but do not contain a Queen?
   (d) contain a three of a kind sequence and 3 values in a row that are not all of the same suit but do not contain a Queen or a black 10?

   Note: the three of a kind and the three value sequence must overlap $4\heartsuit 5\diamondsuit 5\clubsuit 5\spadesuit 6\clubsuit$.