NOTES ON OCT 9, 2012

MIKE ZABROCKI

Exercises: I said that I would solve problems that people asked me about in class. I am going to put the solutions to these (the ones that people asked about) so that you have a reasonable idea of my expectations of what I would like to see as justification of these problems.

(7) $\sum_{n\geq 0}\binom{n+2}{2}\binom{n-2}{2}x^n$

(1) How many ways are there making change for $n = \$1.00$ with pennies, nickels, dimes and quarters such that:
  (a) there are an even number of nickels and no pennies ?
  (b) such that there at most 6 nickels ?
  (c) the total number of nickels and dimes is even ?
  (d) the total number of pennies, dimes and quarters is even ? (*)
(2) How many ways are there of placing $n = 50$ balls in 10 distinguished boxes such that:
  (c) the first 4 boxes have at most 10 of the balls ?
  (d) the first 4 boxes have at least half of the balls ? (*)
(3) (b) Find the generating function for the sequence $a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6, \ldots$ in terms of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.
  (d) Given $D_0 = 1$, $D_1 = a$ and $D_{n+1} = aD_n + bD_{n-1}$ where a, b are unknowns. The entry sequence D_n will be a polynomial in a and b. Find the coefficient of $a^r b^s$.

(*) I will not be able to do (1) (d) and (2) (d) here. I will post a solution at a later date.

(7) I want to give an expression for $\sum_{n\geq 0}\binom{n+2}{2}\binom{n-2}{2}x^n$. I will use one fact that we derived on Sept 27 notes, that if $A(x) = \sum_{n\geq 0} a_n x^n$, then $\frac{x^k}{k!}A^{(k)}(x) = \sum_{n\geq 0}\binom{n}{k}a_n x^n$. I will also use the fact that $x^r A(x) = \sum_{n\geq 0} a_n x^{n+r} = \sum_{n\geq r} a_{n-r}x^n$. Start with $A(x) = \frac{1}{(1-x)^3} = \sum_{n\geq 0}\binom{n+2}{2}x^n$. Notice that since $\binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$ then what I would like to do is decrease the exponent of the x in $\binom{n+2}{2}x^n$ in $A(x)$ by 2 (to $n-2$) then differentiate twice and then multiply until we have the right exponent.

$$A(x) - 1 - 3x = \sum_{n \geq 2} \binom{n+2}{2} x^n$$

$$x^{-2}(A(x) - 1 - 3x) = \sum_{n \geq 2} \binom{n+2}{2} x^{n-2}$$

$$\frac{d^2}{dx^2}\left(x^{-2}(A(x) - 1 - 3x)\right) = \sum_{n \geq 2} \binom{n+2}{2} \frac{d^2}{dx^2}(x^{n-2}) = \sum_{n \geq 2} \binom{n+2}{2} (n-2)(n-3)x^{n-4}$$

$$\frac{1}{2}\frac{d^2}{dx^2}\left(x^{-2}(A(x) - 1 - 3x)\right) = \sum_{n \geq 2} \binom{n+2}{2} \frac{(n-2)(n-3)}{2} x^{n-4}$$

$$x^4 \frac{1}{2}\frac{d^2}{dx^2}\left(x^{-2}(A(x) - 1 - 3x)\right) = \sum_{n \geq 2} \binom{n+2}{2}\binom{n-2}{2} x^n = \sum_{n \geq 0} \binom{n+2}{2}\binom{n-2}{2} x^n$$

(1) (a) Every way of making change for n cents using an even number of nickels and some dimes and quarters is a tuple of the form (X, Y, Z) where X is some means of making change for r cents with an even number of nickels, Y is some means of making change for s cents using dimes and Z is some means of making change for $n-r-s$ cents with quarters. Therefore we have that the generating function for making change with an even number of nickels, and some dimes and quarters is equal to

= (g.f. for the ways of making change using an even number of nickels)(g.f. for

the ways of making change using dimes)(g.f. for the ways of making change using quarters)

If I am using an even number of nickels, then I can make change for n cents in one way if and only if n is a multiple of 10. Therefore

$$\text{g.f. for the ways of making change using an even number of nickels} = \frac{1}{1-x^{10}}$$

Similarly, if I am making change for n cents using dimes , then I can make change for n cents in one way if and only if n is a multiple of 10. Therefore

$$\text{g.f. for the ways of making change using dimes} = \frac{1}{1-x^{10}}$$

If I am making change for n cents using quarters, then I can make change for n cents in one way if and only if n is a multiple of 25.

$$\text{g.f. for the ways of making change using quarters} = \frac{1}{1-x^{25}}$$

We conclude that the g.f. for the number of ways of making change for n cents using an even number of nickels and some dimes and quarters is

$$\frac{1}{(1-x^{10})^2(1-x^{25})}$$

Ideally we would like to take the coefficient of x^{100} and in this case the generating function is simple enough that we can do this by hand. Since $\frac{1}{1-x^{25}} = 1+x^{25}+x^{50}+x^{75}+x^{100}+\cdots$ then

$$\frac{1}{(1-x^{10})^2(1-x^{25})}\Big|_{x^{100}} = \frac{1}{(1-x^{10})^2}\Big|_{x^{100}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{75}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{50}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{25}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{0}}$$

Well we know that there is no way of getting a power of x^{25} or x^{75} in $1/(1-x^{10})^2$ because the only powers that appear are multiples of 10. Moreover we also know that $1/(1-x^{10})^2 = 1+2x^{10}+3x^{20}+4x^{30}+5x^{40}+6x^{50}+\cdots$. Therefore,

$$\frac{1}{(1-x^{10})^2(1-x^{25})}\Big|_{x^{100}} = 11+6+1 = 18\ .$$

(1) (b) Every way of making change with pennies, dimes, quarters and 6 nickels can be broken down into four steps consisting of a way of making change in pennies for r cents, a way of making change in dimes for s cents, a way of making change with quarters for t cents and with at most 6 nickels for $n-r-s-t$ cents. By the multiplication principle of generating functions, we know that

g.f. for making change for n cents with pennies, dimes, quarters and at most 6 nickels

= (g.f. for making change for n cents with pennies)(the ways of making change using dimes)(g.f. for the ways of making change using quarters)(the ways of making change using at most 6 nickels)

The generating functions for making change with pennies, dimes and quarters are similar to the last problem and are respectively $\frac{1}{1-x}$, $\frac{1}{1-x^{10}}$, $\frac{1}{1-x^{25}}$. The g.f. for making change using at most 6 nickels is slightly different. Then there is exactly one way of making change for $0, 5, 10, 15, 20, 25, 30$ cents using at most 6 nickels hence the generating function is given by

$$1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+x^{30} = \frac{1-x^{35}}{1-x^5}$$

Therefore

g.f. for making change for n cents with pennies, dimes, quarters and at most 6 nickels

$$= \frac{1-x^{35}}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

(1) (c) The number of ways of making change for n cents with pennies, quarters and then an even number of nickels and dimes can be seen a way of making change for r cents using pennies, followed by s cents using quarters, followed by $n-r-s$ cents using an even number of nickels and dimes. Therefore,

g.f. for making change for n cents using pennies, quarters and then an even number of nickels and dimes = (g.f. for making change for n cents using pennies)(g.f. for making change for n cents using quarters)(g.f. for making change for n cents using even number of nickels and dimes)

Now there is one way of making change for n cents using pennies hence the generating function is $\frac{1}{1-x}$.

There is one way of making change for n cents using quarters if and only if n is divisible by 25, hence the generating function for making change for n cents using quarters is $\frac{1}{1-x^{25}}$.

Now to make change for n cents with an even number of nickels and dimes is only possible if n is divisible by 5. In fact, if n is divisible by 20 then there is one way to make change for 0 cents and one more way for each 20 cents more and so the generating function for these terms is $1/(1-x^{20}) = 1 + 2x^{20} + 3x^{40} + 4x^{60} + \cdots$. If $n \equiv 5$ (mod 20), then there are no ways of making change for 5 cents with an even number of coins and for every 20 cents there is one more way so the generating function for these terms is $x^{25}/(1-x^{20})^2 = x^{25} + 2x^{45} + 3x^{65} + 4x^{85} + \cdots$. If $n \equiv 10$ (mod 20) then there is one way for making change for 10 cents (2 nickels) and one more way for each 20 cents, so the generating function for these terms is $x^{10}/(1-x^{20})^2 = x^{10} + 2x^{30} + 3x^{50} + 4x^{70} + \cdots$. If $n \equiv 15$ (mod 20) there is one way of making change for 15 cents (one nickel and one dime) and one more way for each 20 cents after so the generating function for these terms is $x^{15}/(1-x^{20})^2 = x^{15} + 2x^{35} + 3x^{55} + 4x^{75} + \cdots$. Since every multiple of 5 is equivalent to 0, 5, 10 or 15 (mod 20) then the generating function for the number of ways of making change for n cents using an even number of nickels and dimes is equal to the sum of the generating functions for n equivalent to 0, 5, 10, or 15 (mod 20), therefore the generating function is equal to $(1 + x^{10} + x^{15} + x^{25})/(1-x^{20})^2$.

We conclude that

g.f. for making change for n cents using pennies, quarters and then an even number

$$\text{of nickels and dimes} = \frac{1 + x^{10} + x^{15} + x^{25}}{(1-x)(1-x^{25})(1-x^{20})^2}$$

(3) (b) We know from class Sept 27 that if $A(x) = \sum_{n\geq 0} a_n x^n$, then

$$x\frac{1}{2}\left(A(x) + A(-x)\right) = a_0 x + a_2 x^3 + a_4 x^5 + a_6 x^7 + \cdots$$

and the generating function for the odd terms (shifted) is

$$\frac{1}{2x}\left(A(x) - A(-x)\right) = a_1 + a_3 x^2 + a_5 x^4 + a_7 x^6 + \cdots$$

hence

$$\frac{x}{2}\left(A(x) + A(-x)\right) + \frac{1}{2x}\left(A(x) - A(-x)\right) = a_1 + a_0 x + a_3 x^2 + a_2 x^3 + a_5 x^4 + a_4 x^5 + a_7 x^6 + a_6 x^7 + \cdots$$

(3)(d) Since $D_0 = 1$ and $D_1 = a$, then $D(x) = \sum_{n\geq 0} D_n x^n = 1 + ax + \sum_{n\geq 2} D_n x^n$, now use the recursive definition for $n \geq 2$, so that

$$D(x) = 1 + ax + \sum_{n\geq 2}(aD_{n-1} + bD_{n-2})x^n = 1 + axD(x) + bx^2 D(x)$$

Solving for $D(x)$, we have $D(x) - axD(x) - bx^2 D(x) = D(x)(1 - ax - bx^2) = 1$, so then

$$D(x) = \frac{1}{1 - ax - bx^2} \ .$$

Now you are asked what is the coefficient of $a^r b^s$ in the coefficient of x^n. To do this we expand as a series in x, then look at the resulting expression and take the coefficient of $a^r b^s$.

$$\frac{1}{1 - ax - bx^2} = \frac{1}{1 - (ax + bx^2)} = \sum_{m\geq 0}(ax+bx^2)^m = \sum_{m\geq 0}(a+bx)^m x^m = \sum_{m\geq 0}\sum_{k\geq 0}\binom{m}{k} a^{m-k} b^k x^{m+k} \ .$$

Now the coefficient of x^n forces $n = m + k$, hence

$$D_n = \sum_{k\geq 0}\binom{n-k}{k} a^{n-2k} b^k.$$

The coefficient of $a^r b^s$ is equal to 0 unless $k = s$ and $n - 2k = r$ (or $n = r + 2s$) and if $n = r + 2s$ then the coefficient is equal to $\binom{r+s}{s}$.