FINAL - TAKE HOME - MATH 4160

ASSIGNED: DECEMBER 4, 2014 DUE: DECEMBER 11, 2014 AT 2:30PM

Write your solutions neatly and clearly. Provide full explanations and justify all of your answers. DO NOT DISCUSS THESE PROBLEMS WITH OTHERS (besides me). You must do this work alone and I will ask you to sign the statement below which states that you have not discussed these problems with others or received help on these problems (when you hand the paper to me). Note that in certain circumstances I am giving you the answer, and it is your job *explain* it. This means that you should write grammatically correct sentences, tell me why two things are equal, and make your calculations clear and easy to follow.

If you have any questions about the problems you may e-mail me at `zabrocki@mathstat.yorku.ca`.

1 (8pts)	.
2 (8pts)	
3 (6pts)	
4 (5pts)	
total(26pts)	

(1) Take S to be a subset of $\{1, 2, \ldots, k\}$ and let A_S represent the number of ways of putting n balls labelled with $\{1, 2, \ldots, n\}$ into $|S|$ boxes which are labeled with the values of S. Let B_S represent the number of ways of putting n balls labelled with $\{1, 2, \ldots, n\}$ into $|S|$ boxes which are labeled with the values of S such that there is at least one ball in each box.

(a) Explain why $A_S = |S|^n$.

(b) Explain why $B_S = |S|!S(n, |S|)$.

(c) Explain why

$$A_S = \sum_{T \subseteq S} B_T \ .$$

(d) Use the principle of inclusion exclusion to show for $n > 0$ and $k \geq 1$,

$$B_{\{1,2,3,\ldots,k\}} = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} r^n$$

(and as a consequence of part (b), $S(n, k) = \frac{1}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} r^n$).

(2) Let $R(n, k)$ represent the number of set partitions of $\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$ into k disjoint non-empty subsets $V_1, \ldots, V_k$ such that, for each $1 \leq j \leq k$, if i is the least integer such that either i or i' belongs to V_j then $\{i, i'\} \subseteq V_j$.

(a) Explain why $R(n, k) = R(n-1, k-1) + k^2 R(n-1, k)$ for $1 < k \leq n$, and $R(n, 1) = 1$. Explain as well why $R(n, 0) = 0$ and $R(n, k) = 0$ for $k > n$.

(b) Calculate a table of values $R(n, k)$ for $1 \leq n \leq 5$.

(c) Define $\langle x\rangle_1 = x$ and for $k > 1$, $\langle x\rangle_k = (x-(k-1)^2)\langle x\rangle_{k-1}$. Calculate $\langle x\rangle_3, \langle x\rangle_4, \langle x\rangle_5$. Notice that $x = \langle x\rangle_1$ and $x^2 = \langle x\rangle_2 + \langle x\rangle_1$. Calculate an expansion of x^3, x^4 and x^5 in terms of $\langle x\rangle_r$ for $r \geq 1$.

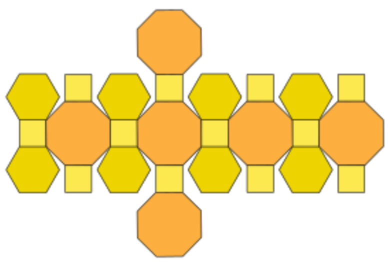
(d) Prove for $n \geq 1$,

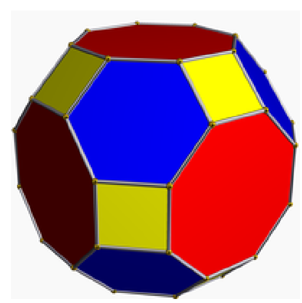
$$x^n = \sum_{k=1}^{n} R(n,k)\langle x\rangle_k \ .$$

(e) (bonus) Show that

$$cosh(\sqrt{2u(cosh(x)-1)}) = \sum_{n\geq 0}\sum_{k\geq 0} R(n,k)\frac{x^{2n}}{(2n)!}u^k \ .$$

(3) Consider the ways of coloring the truncated cuboctahedron `http://en.wikipedia.org/wiki/Truncated_cuboctahedron`. The truncated cuboctahedron has 12 square faces, 8 hexagonal faces and 6 octagonal faces. I have taken the pictures from the wikipedia page (the one on the left is called the *net* of the polyhedron) that you can use, but it might be helpful but you should refer directly to that page for details about the shape.





(a) Give a one (or two) sentence(s) which explains how many rotations and reflections there are in the group of motions and reflections of this object.

(b) List the group of motions and reflections in cycle notation acting on the faces (label the faces on the net). You may just list representatives of the group with a given cycle type and how many there with the same cycle structure.

(c) Count the number of different colorings of the object under the group of motions and reflections using one red face, 13 black faces and 12 white faces.

(4) Find the number of ways of coloring a necklace with 6 beads using k colors for the beads such that no two adjacent beads have the same color and you are allowed to turn the necklace over as well as rotate the beads around the necklace. Be clear about what group you used in your computation.

Notes:

(1) For all of your calculations I do not mind if you use the computer, however I expect you to explain what it is you are doing. It helps to just copy and paste the calculations from Maple.

(2) In question (2) (e), one way to prove this is to show that both the left hand side and the right hand side satisfy the same differential equation, but you have to figure out what the differential equation is based on the recursion the $R(n,k)$ satisfy.
(3) In question (3) I am asking you to find the group of motions and reflections. This group is larger than just the group of rotations.
(4) In question (4), I don't want to give you a hint by telling you how many there are for a given number of colors. Instead, I will say that you can list how many necklaces there are for $k = 3$ if you work hard to verify your answer. I will verify your answer if you email how many necklaces you found for $k = 4$.
(5) In all questions that use the word 'explain' I am looking for a description of why one should apply the addition or multiplication principle.

WHEN YOU SUBMIT THIS EXAM please sign the following statement and fill out the information below.

I attest that I have completed this exam myself without help from anyone else and I have not discussed the problems on this exam with other students in the class.

____

This exam is open book, open notes, and other sources, but I expect you to not ask other people how to complete the assignment. Everyone should list books and websites that you consulted below. If you cannot sign the above statement truthfully, I would prefer if you just explain to me the situation rather than perjure yourself. Please detail below the sources you consulted, the that you have obtained on this exam or who you have discussed these problems with: