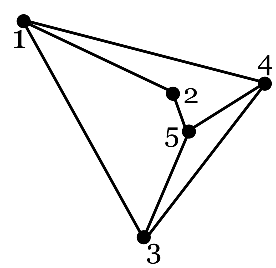
HOMEWORK #3 - MATH 4160

ASSIGNED: OCT 30, 2014 DUE: NOV 11, 2013

Write your homework solutions neatly and clearly (type!). Provide full explanations and justify all of your answers.

(1) Give an argument which explains why the number of ways of coloring the vertices of the following graph with k colors such that no two vertices connected with an edge is colored the same, is equal to $k^5 - 7k^4 + 19k^3 - 23k^2 + 10k$ (the number of ways of coloring this graph with no restriction on the colors is k^5). Phrase your explanation in terms of an applications of the addition principle and the multiplication principle.



(2) Find the generating function for the number of words of length n using letters $a, b, c, d, e, f, g, h, i$ such that
  (a) all 9 letters occur without restriction
  (b) at least one of the first 6 letters appears
  (c) the first 6 letters each appear at least once and the last three each appear an even number of times
  (d) the first 6 letters each appear at least once and the last six each appear an even number of times
  (e) at least one of the first 6 letters appears and the total number of the last 6 letters is even

Use your generating function to find the number of number of words of length 20 with the restrictions above.

As a hint, for length 10 the number of words for part (a) is 3486784401, (b) 3486725352, (c) 45465840, (d) 680400, (e) 1743362676.

(3) In class we found the exponential generating function for the Bell numbers $B(n)$ which are defined by the recurrence $B(0) = 1$, $B(1) = 1$ and $B(n+1) = \sum_{i=1}^{n} \binom{n}{i} B(n-i)$ for all $n \geq 1$. We found that $B(x) = \sum_{n\geq 0} B(n) \frac{x^n}{n!} = e^{e^x - 1}$. Recall that the Stirling numbers of the second kind $S_{n,k}$ are defined as the number of set partitions into k parts. They are defined recursively as $S_{0,0} = 1$, $S_{n,1} = S_{n,n} = 1$ for all $n \geq 1$, and $S_{n,k} = 0$ if $k > n$. Moreover $S_{n+1,k} = kS_{n,k} + S_{n,k-1}$ for $n \geq 0$ and $1 \leq k \leq n$.

Refine the computation that gives the formula for $B(x) = \sum_{n\geq 0} B(n)\frac{x^n}{n!} = e^{e^x-1}$ to show that

$$S(x,q) = \sum_{n\geq 0}\sum_{k\geq 0} S_{n,k} q^k \frac{x^n}{n!} = e^{q(e^x-1)}.$$

(4) Use the result of the previous problem to give the generating function for the number of set partitions into an odd number of parts. That is, if we let

$$B_o(n) = \sum_{k=0}^{\lceil n/2\rceil -1} S_{n,2k+1},$$

then find a formula for $B_o(x) = \sum_{n\geq 0} B_o(n)\frac{x^n}{n!}$.

(5) Given the generating function $A(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$, find a formula for the generating function

$$\tilde{A}(x) = a_1 + a_0 x + a_3 x^2 + a_2 x^3 + a_5 x^4 + a_4 x^5 + \cdots$$

If $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!} = b_0 + b_1\frac{x}{1!} + b_2\frac{x^2}{2!} + \cdots$ is an exponential generating function, find a formula for the generating function

$$\tilde{B}(x) = b_1 + b_0\frac{x}{1!} + b_3\frac{x^2}{2!} + b_2\frac{x^3}{3!} + b_5\frac{x^4}{4!} + b_4\frac{x^5}{5!} + \cdots$$