NOTES FROM THE SECOND CLASS

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In the first class we discussed three tools. Let me restate them again here (in a more general form).

(1) the equality principle:
If there is a bijection between a finite set A and a finite set B, then they have the same number of elements.

(2) the addition principle:
Say there are sets $A_1, A_2, \dots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \cup A_2 \cup \cdots \cup A_n$ is

$$a_1 + a_2 + a_3 + \cdots + a_n$$

(3) multiplication principle
say there are sets $A_1, A_2, \dots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \dots, x_n)$ where $x_i \in A_i\}$ is $a_1 a_2 \cdots a_n$

Application:
$S(n,k)$ = the number of set partitions of $\{1, 2, \dots, n\}$ into k subsets
E.g.

$$\{123\}$$
$$\{12,3\}, \{13,2\}, \{1,23\}$$
$$\{1,2,3\}$$

$$\{1234\}$$
$$\{123,4\}, \{124,3\}, \{134,2\}, \{234,1\}, \{12,34\}, \{13,24\}, \{14,23\}$$
$$\{12,3,4\}, \{13,2,4\}, \{14,2,3\}, \{23,1,4\}, \{24,1,3\}, \{34,1,2\}$$
$$\{1,2,3,4\}$$

1			
1	1		
1	3	1	
1	7	6	1

but I can't do more of this table by hand because it there are too many set partitions of 5.

So let me argue the following using the three principles we start this class with.

All set partitions of $\{1, 2, \ldots, n\}$ into k parts = the set partitions where n is by itself into $k-1$ other parts union the set partitions where n is with one of the other k parts of $\{1, 2, , n-1\}$ so

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \ .$$

This allows us to compute the table of values of $S(n, k)$ much further than we did before without actually counting each individual one.

$$\begin{array}{cccccc} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 7 & 6 & 1 \\ 1 & 15 & 25 & 10 & 1 \\ 1 & 31 & 90 & 65 & 15 & 1 \\ \vdots \end{array}$$

What we would like to do is start with

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$

and then to generalize this and get to

$$1^r + 2^r + \cdots + n^r = ???$$

Just to show what we are up against:

$$1 + 2 + 3 + \cdots + n = n(n+1)/2 \tag{1}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6 \tag{2}$$

$$1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4 \tag{3}$$

$$1^4 + 2^4 + \cdots + n^4 = ??? \tag{4}$$

I don't even know what the right hand side is for the last of these equations.

I showed a technique for demonstrating equalities like the one above, but this technique only works if you know the right hand side. I showed the following general trick called 'telescoping sums.'

In order to show that

$$a(1) + a(2) + \cdots + a(n) = b(n)$$

for some formulas $a(n)$ and $b(n)$ and $b(0) = 0$, then all you need to do is show that $b(n) - b(n-1) = a(n)$. If you do then

$$b(n) - b(n-1) = a(n)$$

$$b(n-1) - b(n-2) = a(n-1)$$

$$\vdots$$
$$b(2) - b(1) = a(2)$$
$$b(1) - b(0) = a(1)$$

Now add up all the terms on the left hand side and we have

$$(b(n) - b(n-1)) + (b(n-1) - b(n-2)) + \cdots + (b(1) - b(0)) = b(n) - b(0) = b(n)$$

If you add up all the terms on the right hand side of the equality then you have

$$a(n) + a(n-1) + \cdots + a(2) + a(1)$$

and they must be equal.

But there is a sequence of equations that continues (unlike equations (1)-(4)):

$$1 + 2 + 3 + \cdots + n = n(n+1)/2(\textbf{??}) \tag{5}$$

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = (n+1)n(n-1)/3 \tag{6}$$

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)/4 \tag{7}$$

$$\vdots$$

$$1 \cdot 2 \cdots k + 2 \cdot 3 \cdots (k+1) + \cdots + n \cdot (n+1) \cdots (n+k-1) = n \cdot (n+1) \cdots (n+k)/(k+1) \tag{8}$$

You should be able to prove this entire sequence of equations either by (a) induction (on n) or (b) telescoping sums.

By telescoping sums, you need only do the computation,

$$\frac{1}{k+1} n(n+1)(n+2) \cdots (n+k) - \frac{1}{k+1}(n-1)n(n+1) \cdots (n+k-1) =$$
$$\frac{1}{k+1}((n+k) - (n-1))n(n+1) \cdots (n+k-2) = n(n+1) \cdots (n+k-1)$$

Therefore, by the method of telescoping sums, (8) follows and all the equations (**??**)-(8) are special cases of this one.

Define for k and integer with $k > 0$, set:

$$(x)^{(k)} = x(x+1)(x+2) \cdots (x+k-1)$$

such that there are k terms in the product.

Examples: $(x)^{(1)} = x$, $(x)^{(2)} = x(x+1)$, $(x)^{(3)} = x(x+1)(x+2), \ldots$

This is new notation that makes some of our formulas simpler. Equations (**??**) - (8) are now

$$(1)^{(1)} + (2)^{(1)} + \cdots + (n)^{(1)} = \frac{(n)^{(2)}}{2} \tag{9}$$

$$(1)^{(1)} + (2)^{(2)} + \cdots + (n)^{(2)} = \frac{(n)^{(3)}}{3} \tag{10}$$

$$(1)^{(3)} + (2)^{(3)} + \cdots + (n)^{(3)} = \frac{(n)^{(4)}}{4} \tag{11}$$

$$\vdots$$

$$(1)^{(k)} + (2)^{(k)} + \cdots + (n)^{(k)} = \frac{(n)^{(k+1)}}{k+1} \tag{12}$$

Now it arises that the table of numbers $S(n,k)$ appear in the expansion of x^n in terms of $(x)_k$. In particular we have

$$x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)} \tag{13}$$

Example:

$$\begin{aligned}
(x)^{(1)} &= x^1 \\
-(x)^{(1)} + (x)^{(2)} &= -x + x(x+1) = -x + x^2 + x = x^2 \\
(x)^{(1)} - 3(x)^{(2)} + (x)^{(3)} &= x - 3(x^2+x) + (x^3 + 3x^2 + 2x) = x^3 \\
-(x)^{(1)} + 7(x)^{(2)} - 6(x)^{(3)} + (x)^{(4)} &= -x + 7x(x+1) - 6x(x+1)(x+2) + x(x+1)(x+2)(x+3) \\
&= -x + 7(x^2+x) - 6(x^3+3x^2+2x) + x^4 + 6x^3 + 11x^2 + 6x \\
&= x^4
\end{aligned}$$

So it should seem surprising that it is even possible to give a formula for x^n in terms of $(x)^{(k)}$, and hopefully it is even more surprising that these coefficients are counted by combinatorial objects called set partitions.