NOTES FROM THE FIRST TWO CLASSES

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main idea of this class

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$

to

$$1^r + 2^r + \cdots + n^r = ???$$

Just to show what we are up against:

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$
$$1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4$$
$$1^4 + 2^4 + \cdots + n^4 = ???$$

but there is a sequence that continues:

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$

$$1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + \cdots + n(n-1) = (n+1)n(n-1)/3$$

$$1 \cdot 0 \cdot (-1) + 2 \cdot 1 \cdot 0 + 3 \cdot 2 \cdot 1 + \cdots + n(n-1)(n-2) = (n+1)n(n-1)(n-2)/4$$

$$\vdots$$

$$1{\cdot}0{\cdot}(-1)\cdots(1-k) + 2{\cdot}1{\cdot}0\cdots(2-k) + \cdots + n{\cdot}(n-1){\cdot}(n-2)\cdots(n-k) = (n+1)n(n-1)\cdots(n-k)/(k+2)$$

Proof either by (a) induction (b) telescoping sums

First class (a) the equality principle

If there is a bijection between a finite set A and a finite set B, then they have the same number of elements.

(b) the addition principle

say there are sets $A_1, A_2, \ldots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \cup A_2 \cup \cdots \cup A_n$ is

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Example: Consider the set of words in 1 and 0 with three 1s and three 0s.

And paths in a 3×3

What about?

$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$

(c) multiplication principle

say there are sets $A_1, A_2, \ldots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \text{ where } x_i \in A_i\}$

is $a_1 a_2 \cdots a_n$

Example: lets say I was going to make a cereal with colored shape marshmellows
colors $= \{pink, yellow, orange, green, purple, red\}$
shapes $= \{hearts, moons, stars, clovers, horseshoes, balloons, pots\}$

I shouldn't have to list all possible marshmellows, $\{$*pink heart*, *pink moons*, *pink stars*, $\ldots$, *red pots*$\}$ instead it is much easier to say that there are 6 colors and 7 shapes so there are $6 \cdot 7 = 42$ marshmellows possible.

flavors $= \{chocolate, strawberry, peanutbutter\}$
eat it with $= \{fork, knife, spoon, chopsticks\}$
Then I could eat *chocolate purple ballons with a fork* (for example) but there should be $|colors| \cdot |shapes| \cdot |flavors| \cdot |eat\ it\ with| = 6 \cdot 7 \cdot 3 \cdot 4$ possibilities.

(d) division and subtraction - much harder, avoid doing it.

Application:

$S(n,k) =$ the number of set partitions of $\{1, 2, \ldots, n\}$ into k subsets

E.g.

$$\{123\}$$
$$\{12,3\}, \{13,2\}, \{1,23\}$$
$$\{1,2,3\}$$

$$\{1234\}$$
$$\{123,4\}, \{124,3\}, \{134,2\}, \{234,1\}, \{12,34\}, \{13,24\}, \{14,23\}$$
$$\{12,3,4\}, \{13,2,4\}, \{14,2,3\}, \{23,1,4\}, \{24,1,3\}, \{34,1,2\}$$
$$\{1,2,3,4\}$$

1			
1	1		
1	3	1	
1	7	6	1

but I can't do more of this table by hand because it there are too many set partitions of 5.

argue:

all set partitions of $\{1, 2, \ldots, n\}$ into k parts = the set partitions where n is by itself into $k-1$ other parts union the set partitions where n is with one of the other k parts of $\{1, 2, , n-1\}$ so

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

1					
1	1				
1	3	1			
1	7	6	1		
1	15	25	10	1	
1	31	90	65	15	1

$\cdots$

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first class: I covered

(1)

$$1{\cdot}0{\cdot}(-1) \cdots (1-k) + 2{\cdot}1{\cdot}0 \cdots (2-k) + \cdots + n{\cdot}(n-1){\cdot}(n-2) \cdots (n-k) = (n+1)n(n-1) \cdots (n-k)/(k+2)$$

(2)

addition and multiplication principle

(3)

definitions of $S(n, k)$ = the number of set partitions of $\{1, 2, \ldots, n\}$ into k parts. A set partition of $\{1, 2, \ldots, n\}$ is a division of $\{1, 2, \ldots, n\}$ into k nonempty and non-intersecting subsets

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \tag{1}$$

for $n > 1$ and $1 \leq k \leq n$ with the convention that $S(n-1, n) = 0$ and $S(n, 0) = 0$.

Proof. For shorthand, let $[n] := \{1, 2, ..., n\}$. The set partitions of $[n]$ into k parts can be divided into two sets, those that have n in a part by itself and those that have n in a part with other values from $[n-1]$. By the addition principle we have

$$S(n, k) = \text{ \# set partitions with } n \text{ in a set alone } + \text{ \# set partitions where } n \text{ is not alone}$$

The number of set partitions of $[n]$ into k parts with n in a part by itself is isomorphic to the set of set partitions of $[n-1]$ into $k-1$ parts by throwing away the set containing just n. This means that the number of set partitions of $[n]$ into k parts with n in a set all by itself is $S(n-1, k-1)$.

For a set partition P of $[n]$ with k parts and n is in a part with other elements, then let x be a value between 1 and k that indicates which of the k parts n is contained in and P' be the set partition of $[n-1]$ into k parts that is formed by removing n from P. Clearly if we know (x, P') then it is possible to recover P, and if we know P it is possible to recover both x and P'. Hence, there are the same number of these objects. Since there are k possible values of x and there are $S(n-1,k)$ possible set partitions P', then there are in total $kS(n-1,k)$ possible set partitions of $[n]$ into k parts where n is not in a part by itself.

Therefore (1) holds true. $\square$

This recursion allows us to compute more of the table than before.

1					
1	1				
1	3	1			
1	7	6	1		
1	15	25	10	1	
1	31	90	65	15	1

$\cdots$

There is an application for set partitions in terms of algebra.

Define for k and integer with $k > 0$, set:

$$(x)_k = x(x-1)(x-2)\cdots(x-k+1)$$

such that there are k terms in the product.

Examples: $(x)_1 = x$, $(x)_2 = x(x-1)$, $(x)_3 = x(x-1)(x-2), \ldots$

This is new notation that makes some of our formulas simpler.

Example: Remember the identity that we

$$1\cdot 0\cdot(-1)\cdots(1-k)+2\cdot 1\cdot 0\cdots(2-k)+\cdots+n\cdot(n-1)\cdot(n-2)\cdots(n-k) = (n+1)n(n-1)\cdots(n-k)/(k+2)$$

which is kind of horrible notation is equivalent to

$$(1)_{k+1} + (2)_{k+1} + \cdots + (n)_{k+1} = (n+1)_{k+2}/(k+2)$$

Now it arises that the table of numbers $S(n,k)$ appear in the expansion of x^n in terms of $(x)_k$. In particular we have

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k \tag{2}$$

Example:

$$(x)_1 = x^1$$

$$(x)_1 + (x)_2 = x + x(x-1) = x + x^2 - x = x^2$$

$$(x)_1 + 3(x)_2 + (x)_3 = x(x-1)(x-2) + 3x(x-1) + x = x^3$$

$$\begin{aligned}(x)_1 + 7(x)_2 + 6(x)_3 + (x)_4 &= x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3)\\ &= x + 7(x^2-x) + 6(x^3-3x^2+2x) + x^4 - 6x^3 + 11x^2 - 6x\\ &= x^4\end{aligned}$$

So it should seem surprising that it is even possible to give a formula for x^n in terms of $(x)_k$, and hopefully it is even more surprising that these coefficients are counted by combinatorial objects called set partitions.

Here is the quick proof that this formula holds:

Proof. We will prove this by induction on n. We have already shown the base case for $n = 1, 2, 3, 4$ above.

Assume that (2) holds for some fixed n. Then we have

$$x^{n+1} = x^n \cdot x = \sum_{k=1}^{n} S(n,k)(x)_k \cdot x \tag{3}$$

$$= \sum_{k=1}^{n} S(n,k)(x)_k(x-k+k) \tag{4}$$

$$= \sum_{k=1}^{n} S(n,k)(x)_k(x-k) + \sum_{k=1}^{n} kS(n,k)(x)_k \tag{5}$$

$$= \sum_{k=1}^{n} S(n,k)(x)_{k+1} + \sum_{k=1}^{n} kS(n,k)(x)_k \tag{6}$$

$$= \sum_{k=2}^{n+1} S(n,k-1)(x)_k + \sum_{k=1}^{n} kS(n,k)(x)_k \tag{7}$$

$$= S(n,n)(x)_{n+1} + \sum_{k=2}^{n} S(n,k-1)(x)_k + \sum_{k=2}^{n} kS(n,k)(x)_k + S(n,1)(x)_1 \tag{8}$$

$$= S(n,n)(x)_{n+1} + \sum_{k=2}^{n} (S(n,k-1) + kS(n,k))(x)_k + S(n,1)(x)_1 \tag{9}$$

$$= S(n,n)(x)_{n+1} + \sum_{k=2}^{n} S(n+1,k)(x)_k + S(n,1)(x)_1 \ . \tag{10}$$

Some comments about this calculation: from step (6) to step (7) we did a shift of indices $k \to k-1$ (but they are the same sum). From step (7) to (8) we broke off the $k = n+1$ term of the first sum and the $k = 1$ term of the second sum. From step (9) to (10) we applied (1) with $n \to n+1$. Now recall that $S(n,n) = S(n+1,n+1) = 1$ and $S(n,1) = S(n+1,1) = 1$, hence we can rewrite the first and last term so that they are consistent with the other terms in this sum and hence we have shown

$$x^{n+1} = \sum_{k=1}^{n+1} S(n+1,k)(x)_k$$

which is equation (2) with $n \to n+1$.

Therefore by the principle of mathematical induction, (2) is true for all $n \geq 1$. □

Because of the equation

$$\sum_{i=1}^{n}(i)_{k+1} = (1)_{k+1} + (2)_{k+1} + \cdots + (n)_{k+1} = (n+1)_{k+2}/(k+2)$$

that we wrote down above, this allows us to sum powers of i^r.

$$\sum_{i=1}^{n} i^1 = \sum_{i=1}^{n}(i)_1 = (n+1)_2/2 = (n+1)n/2$$

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n}((i)_1 + (i)_2) = \sum_{i=1}^{n}(i)_1 + \sum_{i=1}^{n}(i)_2 = (n+1)_2/2 + (n+1)_3/3$$

With a little algebra we can show:

$$(n+1)_2/2+(n+1)_3/3 = (n+1)n/2+(n+1)n(n-1)/3 = (n+1)n(1/2+(n-1)/3) = n(n+1)(2n+1)/6$$

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n}((i)_1 + 3(i)_2 + (i)_3) = (n+1)_2/2 + 3(n+1)_3/3 + (n+1)_4/4$$

The right hand side is a polynomial in n of degree 4 and we can calculate directly that,

$$(n+1)_2/2+3(n+1)_3/3+(n+1)_4/4 = (n+1)n/2+(n+1)n(n-1)+(n+1)n(n-1)(n-2)/4 = n^2(n+1)^2/4$$

And the formula for the sum of the 4^{th} powers of i is

$$\sum_{i=1}^{n} i^4 = \sum_{i=1}^{n}((i)_1 + 7(i)_2 + 6(i)_3 + (i)_4) = (n+1)_2/2 + 7(n+1)_3/3 + 6(n+1)_4/4 + (n+1)_5/5$$

and the right hand side in the form it is in is cleaner than calculating the polynomial:

$$(n+1)_2/2 + 7(n+1)_3/3 + 6(n+1)_4/4 + (n+1)_5/5 = n(n+1)(2n+1)(1-3n+3n^2)/30 \ .$$

What is great about what we have done is here is that it is difficult to conjecture the right hand side of this sum or for higher powers (so that one might prove it by some other means). Instead here we have proven an explicit formula which works for all powers of r, that is:

$$\sum_{i=1}^{n} i^r = \sum_{k=1}^{r} S(r,k)(n+1)_{k+1}/(k+1) \ .$$

NOTES ON SEPT 13-18, 2012

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Last time I gave a name to

- $S(n,k) :=$ number of set partitions of $[n]$ into k parts. This only makes sense for $n \geq 1$ and $1 \leq k \leq n$. For other values we need to choose a convention that makes sense. We also have that $S(n,1) = S(n,n) = 1$ and $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ for $1 < k < n$.

This time I started to add to the sets we can count and give recursive formulas for:

- $P(n) :=$ the number of permutations of n, and by default $P(0) = P(1) = 1$ (I said $n!$, but I thought about it and $n!$ in most places is referring to the algebraic definition, these are the same thing, but its better to reserve the symbol $n!$ for the algebraic definition).
- $\binom{n}{k} :=$ the number of ways of choosing k elements from the set $\{1,2,\ldots,n\}$
- $B(n) :=$ the number of set partitions of $\{1,2,\ldots,n\}$ (and $B(0) = 1$)
- $s'(n,k) :=$ the number of permutations of $\{1,2,\ldots,n\}$ with exactly k cycles (and $s(n,k) = (-1)^{n-k}s'(n,k)$)

$S(n,k)$ are called the Stirling numbers of the second kind

$s(n,k)$ are called the Stirling numbers of the first kind and $s'(n,k)$ are called the (unsigned) Stirling numbers of the first kind.

Remark, that they are related to algebra by the formula (try some examples of these formulas to make sure you are comfortable with them, you are asked to prove the last one as a homework exercise exactly as I had done it in class for the the first one.

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k$$

$$(x)_n = \sum_{k=1}^{n} s(n,k)x^k$$

$$x^n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k)(x)^{(k)}$$

$$(x)^{(n)} = \sum_{k=1}^{n} s'(n,k)x^n$$

Remark: These things are discussed in section 1.4, 1.8, 2.9 and 2.13 of the book, but it would be good to familiarize yourself with the other sections in chapter 1 and 2.

I explained bijectively why for $n > 1$,

$$P(n) = nP(n-1)\ . \tag{1}$$

Conclusion $P(n) = n! = n(n-1)(n-2)\cdots 2\cdot 1$.

I explained bijectively that for $n \geq 1$ and $0 \leq k \leq n$,

$$n! = \binom{n}{k} k!(n-k)! \tag{2}$$

Conclusion $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

I explained that while it was a simple application of the addition principle to state why

$$B(n) = \sum_{k=1}^{n} S(n,k) \tag{3}$$

that we could also argue bijectively that

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(n-k-1) \tag{4}$$

This allowed us to compute $B(n)$ for $1 \leq n \leq 4$ in two different ways:

Table of $S(n,k)$

$n\backslash k$	1	2	3	4	5	6	$B(n)$
1	1						1
2	1	1					2
3	1	3	1				5
4	1	7	6	1			15
5	1	15	25	10	1		52
6	1	31	90	65	15	1	203

also

$$B(2) = \binom{1}{0}B(1) + \binom{1}{1}B(0) = 2$$

$$B(3) = \binom{2}{0}B(2) + \binom{2}{1}B(1) + \binom{2}{2}B(0) = 2+2+1 = 5$$

$$B(4) = \binom{3}{0}B(3) + \binom{3}{1}B(2) + \binom{3}{2}B(1) + \binom{3}{3}B(0) = 5 + 3\cdot 2 + 3\cdot 1 + 1 = 15$$

Notice that this second way is not particularly more efficient than calculating with the Stirling numbers, but it does proved a very different sort of formula.

We also computed the first few examples of the Stirling numbers of the first kind (with signs). I used the expansion $(x)_n = \sum_{k=1}^{n} s(n,k)x^k$ (which you are supposed to prove for homework).

$$x(x-1) = x^2 - x$$
$$x(x-1)(x-2) = x^3 - 3x^2 + 2x$$
$$x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$$

Table of $s(n,k)$

$n\backslash k$	1	2	3	4	5	6	$n!$
1	1						1
2	-1	1					2
3	2	-3	1				6
4	-6	11	-6	1			24

We also computed a few of these values by the use of the definition $s(n,k)$.

$s'(4,3) = 6$: this is because the permutations of 1234 into 3 cycles have two fixed points and two elements swapped. Once we know which are the fixed points are then the permutation is determined. So the number of permutations of 1234 into 3 cycles is $\binom{4}{2} = 6 =$ the number of ways of picking two fixed points.

I computed also that $s'(4,2) = 11$ because there are two choices, either there is one fixed point and the remaining three elements are in a cycle or there are two pairs of elements that are being exchanged. In the first case, there are 4 possible fixed points and two possible cycles, so there are 8 permutations with one fixed point and two cycles. In the second case we can choose to swap 1 with 2,3 or 4 and the remaining two of those choices will also be swapped, so there are 3 permutations with two 2-cycles. In total there are 8+3=11 permutations with two cycles.

Stirling numbers and their relationship with polynomials is discussed in section 2.9 and 2.13.

I wanted to give some examples of counting sets of objects like those that were in the homework. One of the best examples of this is counting poker hands. Poker is a card game played with a 52 card deck with 13 values for the cards and 4 suits. Poker hands are ranked by how common a hand is. For instance, there are 13×48 possible 4-of-a-kind hands because we can choose which value appears 4 times in a 4-of-a-kind hand plus one extra card from the remaining 48 cards in the deck. There are also 40 straight flush hands because there are 4 possible suits and 10 possible straights. Therefore a straight flush beats a 4-of-a-kind.

The number of 5 card poker

If we want to count a set of possible hands we need to apply the multiplication principle and the addition principle sometimes in creative ways.

For instance if I want to count the number of hands that have exactly one pair, then I note that every pair is determined by the following 4 peices of information.

- the value of the card that appears twice
- the values of the other three cards (all different and not the same as the last value)
- the two suits used by the pair
- a suit used by the smallest of the three cards
- a suit used by the middle of the three cards
- a suit used by the largest of the three cards

That is I am saying that if I am given a particular five card hand with containing exactly a pair, then the 6 pieces of information are all that is necessary to determine the hand and the hand determines the information. Therefore the set of hands containing a pair are in bijection with tuples containing the information in that list. For example the hand 3$\heartsuit$, 5$\diamondsuit$, 7$\diamondsuit$, 7$\clubsuit$, 10$\spadesuit$ and this is isomorphic to this list $(7, \{3, 5, 10\}, \{\diamondsuit, \clubsuit\}, \heartsuit, \diamondsuit, \spadesuit)$.

Now there are 13 ways of choosing the card that appears twice; $\binom{12}{3}$ ways of choosing a set of three elements from the 12 values that are not the pair; there are $\binom{4}{2}$ possible sets for the suits which appear in the pair; there are 4 suits possible for the non-pair card; 4 suits for the second non-pair card; 4 suits for the third non-pair card. In total there are

$$13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4 \cdot 4 \cdot 4 =$$

is the number of hands with exactly one pair.

I also counted the number of hands with exactly two pairs. The following information completely determines a hand that has a two pair.

- two values (an upper and a lower) which will each appear twice in the hand
- two suits of the 4 for the lower value
- two suits of the 4 for the upper value
- a last card which is any of the $52 - 8$ cards which don't have a value of the pair.

Again, I can frame this in terms of a bijection with a list of information. A hand with 5 cards in it is in bijection with a list containing 4 pieces of information. For instance the hand 3$\heartsuit$, 3$\clubsuit$, 7$\clubsuit$, $K\spadesuit$, $K\clubsuit$ is a hand with two pairs. It is in bijection with $(\{3, K\}, \{\heartsuit, \clubsuit\}, \{\spadesuit, \clubsuit\}, 7\clubsuit)$.

Now the number of possible lists are easy to count by the multiplication principle. There are $\binom{13}{2}$ choices for the values of the pairs. There are $\binom{4}{2}$ possible sets of two suits from the set $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$ and there are 44 remaining cards. Therefore the number of hands with two pairs is

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 \ .$$

I mentioned that you should avoid subtraction if you can, but there are always exceptions to that rule. The reason I would like you to avoid subtraction is that it is hard to explain clearly. One obvious exception to that rule is the number of straight hands which are not straight flushes. By basic counting techniques we know that there are $10 \cdot (4^5 - 4)$ possible straight hands which don't have a flush because there are 10 possible straights (that begin with A through 10 as the lowest card) and there are 4^5 ways of picking a suit for each of the cards of the straight, BUT we have to subtract off the number of ways that all suits are the same. This is the easy way of explaining how to pick the suits and it involves subtraction.

There is the hard way of explaining how to pick the suits too that only involves addition. Let me count the value $4^5 - 4$ in a different way. We know that either the hand contains 2, 3 or 4 different suits.

- Say there are two different suits, a first suit and a second suit. There are $S(5, 2)$ ways of distributing the suits among the 5 cards (for example the set partition $\{\{1, 3\}, \{2, 4, 5\}\}$ means that the straight will have the form $1X$, $2Y$ $3X$, $4Y$, $5Y$ where X is the first suit and Y is the second suit and $1, 2, 3, 4, 5$ will be the values in the straight) and 4 ways of picking the first suit and 3 ways of picking the second suit. There are $S(5, 2) \cdot 4 \cdot 3$ ways of having two suits in your poker hand which is a straight.
- Say that there are 3 different suits that appear in the hand. There are $S(5, 3)$ ways of distributing the suits among the 5 cards, 4 ways of picking the first suit, 3 ways of picking the second suit (can't be the same as the last), 2 choices for the third suit. Thus there are $S(5, 3) \cdot 4 \cdot 3 \cdot 2$ ways of having a straight hand with exactly 3 different suits.
- Say that all 4 suits appear in your hand. Then there are $S(5, 4)$ ways of distributing the suits among the cards in the hand and 4! ways of assigning an order to the suits.

Therefore the number of ways of choosing suits for a hand such that not all suits are the same is $S(5,2)\cdot4\cdot3+S(5,3)\cdot4\cdot3\cdot2+S(5,4)\cdot4\cdot3\cdot2\cdot1 = 15\cdot4\cdot3+25\cdot4\cdot3\cdot2+10\cdot4\cdot3\cdot2\cdot1 = 4^5-4$. This should be compared with the identity that we have already proven

$$4^5 = S(5, 1) \cdot 4 + S(5, 2) \cdot 4 \cdot 3 + S(5, 3) \cdot 4 \cdot 3 \cdot 2 + S(5, 4) \cdot 4 \cdot 3 \cdot 2 \cdot 1 + S(5, 5) \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0$$

This is clearly NOT better than explaining $4^5 - 4$, just different. No matter how you explain why your answer is what it is, you should always get the same value in the end. Computing a value in two different ways gives you a way of checking your answer.

The types of poker hands are:

- royal flush - $10, J, Q, K, A$ all the same suit
- straight flush (sometimes these first two sets are combined): a sequence of 5 cards in order all with the same suit
- 4 of a kind
- flush - five cards all one suit not a straight
- full house - a pair and a three of a kind
- straight - five cards whose values are in a 5 card sequence and it is not the case they all have the same suit
- 3 of a kind
- two pairs
- pair
- none of the above

A really good exercise is to figure out a way of counting the number of each of these sets using only addition (like I said, this is sometimes the more complicated way of coming up with the answer) and then add them all up and check that they add up to $\binom{52}{5}$ (the number of ways of picking 5 cards from a deck of 52).

Counting hands of cards is discussed in section 1.13 in the book.

I also discussed a little about proving combinatorial identities. I don't want an algebraic proof. In the case of the identities that are on the homework, two of the three of them would be completely trivial to show with a little algebra (what I mean is an algebraic proof is just show $n^3 = n(n-1)(n-2) + 3n(n-1) + n$ and you are done with the first one). I talked about in particular

$$n^3 = (n)_1 + 3(n)_2 + (n)_3$$

You are to find a set such that the number of elements in the set is n^3. Here are some ideas (but there are an infinite number of possible answers).

- The number of sequences of three values where there are n choices for each of the three values.
- The number of three digit numbers base n (I then proceeded to count in binary (base 2) on my fingers, and then lost track at 20 and I was still on my right hand).
- The number of ways of painting three different rooms with n different colors

After a while these combinatorial interpretations all begin to sound the same. Then you need to find a combinatorial interpretation for each of the terms on the right hand side and find a bijection to them. For instance $(n)_3 = n(n-1)(n-2)$ is equal to "the number of ways of painting three different rooms with n different colors where all the colors used are different."

Combinatorial identities are discussed in section 2.3 in the book.

NOTES ON SEPT 20 - 25, 2012

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We considered rearranging letters of a word. I looked at the number of rearrangements of the word ANNOTATE. Consider rearrangements of the letters like TNTAAOEN or NEONATAT. I said that the following procedure will determine the word

- pick two positions from 8 for the letter A
- pick one position from the remaining 6 for the letter E
- pick two positions from the remaining 5 for the letter N
- pick one position from the remaining 3 for the letter O

the remaining two positions of the word will be filled with T's. That the set of rearrangements of the word ANNOTATE is in bijection with the sequences of subsets of $\{1, 2, \ldots, 8\}$ consisting of a subset of size 2, a subset of size 1, a subset of size 2 and a subset of size 1.

For example the word TNTAAOEN is sent under this bijection to $(\{4, 5\}, \{7\}, \{2, 8\}, \{6\})$. The number of such sequences is equal to

$$\binom{8}{2}\binom{6}{1}\binom{5}{2}\binom{3}{1} = \frac{8!}{2!6!}\frac{6!}{1!5!}\frac{5!}{2!3!}\frac{3!}{1!2!} = \frac{8!}{2!1!2!1!}$$

For this we define the notation we will call the multi-choose or multinomial coefficient. We will define $\binom{n}{k_1,k_2,\cdots,k_r}$ to be the number of ways of picking subsets of size $k_1, k_2 \ldots, k_r$ from an n element set For a sequence of integers $k_1, k_2, \ldots, k_r \geq 0$ such that $k_1 + k_2 + \cdots + k_r \leq n$, then

$$\begin{aligned}\binom{n}{k_1, k_2, \cdots, k_r} &= \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{r-1}}{k_r} \\ &= \frac{n!}{k_1!k_2!\cdots k_r!(n-k_1-k_2-\cdots-k_r)!}\ .\end{aligned}$$

If $k_1 + k_2 + \cdots + k_r > n$ then $\binom{n}{k_1,k_2,\cdots,k_r} = 0$.

There is another place where this coefficient arises. I assume that everyone is familiar with the binomial theorem which gives an expansion of $(1 + x)^n$ in terms of the binomial coefficients $\binom{n}{k}$. We have

$$(1+x)^n = \sum_{k\geq 0}\binom{n}{k}x^k = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

for example, we have in particular

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 + 0x^5 + 0x^6 + 0x^7 + \ldots$$

The multinomial coefficient is a generalization of these coefficients. In fact, we have

$$(1+x_1+x_2+\cdots+x_r)^n = \sum_{k_1+k_2+\cdots+k_r\leq n} \binom{n}{k_1,k_2,\dots,k_r} x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r}$$

With so many unknowns in this equation it is hard to appreciate this formula. But try an example. I can use the computer and see that $(1+x+y)^4 =$

$$1+4\,x+4\,y+6\,x^2+12\,xy+6\,y^2+4\,x^3+12\,x^2y+12\,xy^2+4\,y^3+x^4+4\,x^3y+6\,x^2y^2+4\,xy^3+y^4$$

I can use this formula to see that $\binom{4}{1,2} = \frac{4!}{1!2!1!} = 12$ and I see that the coefficient of xy^2 in this expression is 12. If I want to answer a question like what is the coefficient of $x^7y^3z^9$ in the expression $(1+x+y+z)^{40}$ then I have a formula for this value, it is $\binom{40}{7,3,9} = \frac{40!}{7!3!9!21!}$ just as the binomial theorem tells me the coefficient of x^{19} in $(1+x)^{40}$ is $\binom{40}{19} = \frac{40!}{19!21!}$.

I also wanted to know *how many* terms there are in the expression $(1+x_1+x_2+\cdots+x_r)^n$ (and not just an expression for the coefficient). In order to do this I looked at the number of monomials of degree d for $d=0,1,2,3,4,5,\dots$ and when $r=1,2,3,4,\dots$. Lets look at examples like

$$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$$
$$(1+x)^6 = 1+6x+15x^2+20x^3+15x^4+6x^5+x^6$$

the number of terms of degree d is always 1 as long as d is less then or equal to the power n in $(1+x)^n$. Now lets try this for $r=2$ variables. We computed $(1+x+y)^4$, so lets look at

$$(1+x+y)^5 = 1+5\,x+5\,y+10\,x^2+20\,xy+10\,y^2+10\,x^3+30\,x^2y+30\,xy^2+10\,y^3+$$
$$5\,x^4+20\,x^3y+30\,x^2y^2+20\,xy^3+5\,y^4+x^5+5\,x^4y+10\,x^3y^2+10\,x^2y^3+5\,xy^4+y^5$$

We see that at degree 0 there is 1 term, degree 1 there are two terms, degree 2 there are 3 terms, degree there are 4 terms, etc. (as long as $d \leq n$).

For three variables, the monomials of degree 1 are 3 monomials x, y, z; at degree 2 there are 6 monomials, $x^2, y^2, z^2, xy, xz, yz$; at degree 3 there are 10 monomials x^3, y^3, z^3, x^2y, x^2z, y^2x, y^2z, z^2x, z^2y, xyz; and one can conjecture that the pattern of the next value increases by 2,3,4,... continues.

If you write down a table, you see that the values we have computed thus far have a familiar pattern.

$r\backslash d$	0	1	2	3	4	5
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4				

If you look closely at this table you see the binomial coefficients (Pascal's triangle). It might cause you to conjecture that the formula for the number of monomials in r variables of degree d is equal to $\binom{d+r-1}{d}$.

How do we explain this? First, notice that the exponents of a monomial can be translated to a list. $x^5y^3z^1w^6$ can be represented by $(5,3,1,6)$ without losing information. The degree of the monomial is equal to the sum of the entries in this list. Therefore we have shown:

# of monomials of degree d with r variables
= # of sequences of r non-negative integers that sum to d

Now there is another transformation that we can do in order to count these lists. Every sequence of non-negative integers can be translated into a sequence of dots $\bullet$ and bars $|$. The sequence $(a_1, a_2, \ldots, a_{r-1}, a_r)$ is in bijection with a_1 dots $\bullet$ followed by a bar $|$, a_2 dots $\bullet$ followed by a bar $|$, $\ldots$, a_{r-1} dots $\bullet$ followed by a bar $|$ followed by a_r dots $\bullet$. Notice that I don't need to finish with the bar because it would always be there so I leave it off. For example the sequence $(3,0,0,1,1,2,0)$ is sent to $\bullet\bullet\bullet|||\bullet|\bullet|\bullet\bullet|$ and we can recover the word consisting of d dots $\bullet$ and $r-1$ bars $|$ from the sequence of integers and we can recover the sequence of integers from the word, hence we have shown that these two things are in bijection. Therefore we have shown

# of monomials of degree d with r variables
= # of sequences of r non-negative integers that sum to d
= # of words with d symbols $\bullet$ and $r-1$ symbols $|$

This last set, we know how to count. In total there are $d+r-1$ letters in our word and d of them are $\bullet$ and $r-1$ of them are $|$, therefore we need only "choose" the subset of d positions where the $\bullet$s belong. Hence this is also in bijection with

# of monomials of degree d with r variables
= # of sequences of r non-negative integers that sum to d
= # of words with d symbols $\bullet$ and $r-1$ symbols $|$
= # subsets of size d of the integers $\{1,2,\ldots,d+r-1\}$

There is also one more set that this is in bijection with that I spent a while explaining. It is the number of ways of choosing d things from a set of size r where you are allowed to have repeat entries. Imagine we have an urn consisting of colored balls with r colors and you reach in and pull out d of them. A subset with repetition is just recording how many you got of each color. "I got 5 blue, 3 red, 1 green and 6 yellow" has the same information as a list of non-negative integers $(5,3,1,6)$. Hence a subset of the set $\{1,2,\ldots,r\}$ of size d where you allow repetitions is also in bijection with this collection of objects and we have:

$\#$ of monomials of degree d with r variables
$= \#$ of sequences of r non-negative integers that sum to d
$= \#$ of words with d symbols $\bullet$ and $r-1$ symbols $|$
$= \#$ subsets of size d of the integers $\{1, 2, \ldots, d+r-1\}$
$= \#$ multi-subsets (repetitions allowed) of size d of the integers $\{1, 2, \ldots, r\}$

I reviewed that we knew how to give a combinatorial interpretation to the binomial coefficient in a couple of odd ways.

Remark 1: How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + \cdots + x_r = n?$$

Answer: $\binom{n+r-1}{n} = \binom{n+r-1}{r-1}$. Why? Think of a dots and bars argument and find a bijection from a solution to this equation represented as a sequence $(x_1, x_2, x_3, \ldots, x_r)$ and a sequence of n dots and $r-1$ bars.

Remark 2: How many paths are there in a lattice grid from $(0,0)$ to (n,m) with n steps $E = (1,0)$ and m steps $N = (0,1)$?
Answer: $\binom{n+m}{n} = \binom{n+m}{m}$. Why? Think of a lattice path in a grid with N and E steps and translate it into a word of letters N and E such that there are m letters N and n letters E. The number of such words is determined by the number of ways of choosing the positions of the E steps in the word.

Now prove

$$\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n} = \binom{2n}{n}$$

Proof 1: (using the combinatorial interpretation in terms of solutions to $x_1+x_2+\cdots+x_r = n$)
Let A be the set of non-negative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_{n+1} = n$$

By Remark 1, we know that there are $\binom{2n}{n}$ such solutions. Let A_k be the subset of the solutions $(x_1, x_2, \ldots, x_n, x_{n+1})$ where $x_{n+1} = n-k$, then it must be that $x_1+x_2+\cdots+x_n = k$ and so also by Remark 1 we know that $|A_k| = \binom{n+k-1}{k}$ such solutions. Since

$0 \leq x_{n+1} \leq n$, we know that

$$A = \bigcup_{k=0}^{n} A_k$$

hence

$$\binom{2n}{n} = |A| = \sum_{k=0}^{n} |A_k| = \sum_{k=0}^{n} \binom{n+k-1}{k} .$$

Proof 2: (using the path combinatorial interpretation) Let A be the set of paths from $(0,0)$ to (n,n) using N and E steps. By Remark 2, there are $\binom{2n}{n}$ such paths. Let A_k be the set of paths in A such that the last horizontal step is at height k. A path in the set A_k consists of a path from $(0,0)$ to $(n-1,k)$ followed by a horizontal step to (n,k) followed by $n-k$ vertical steps. By Remark 2, there are $\binom{n+k-1}{k}$ paths from $(0,0)$ to $(n-1,k)$ and these paths determine completely the rest of the path, then $|A_k| = \binom{n+k-1}{k}$. Since $A = \bigcup_{k=0}^{n} A_k$,

$$\binom{2n}{n} = |A| = \sum_{k=0}^{n} |A_k| = \sum_{k=0}^{n} \binom{n+k-1}{k} .$$

I also found this example that I liked in the exercises in the book. Show

$$x^n - 1 = (x-1) + (x-1)x + (x-1)x^2 + \cdots + (x-1)x^{n-1} .$$

Proof: First start by assuming that $n, x \geq 1$ (otherwise the following argument won't make much sense). Let x represent the number of colors (one of which will be red), and n be a collection of n (ordered/distinct) rooms. Let A be the set of colorings of the n rooms with x colors such that all of the rooms are not red. There are x^n ways of coloring the rooms with no restriction on the colors since for each of the n rooms there is a choice of x colors. Therefor $|A| = x^n - 1$ since we exclude the possibility that all rooms are colored red. Now for $1 \leq k \leq n$, let A_k be the set of colorings of rooms in A such that the first room which is not red is the $n-k+1^{st}$ room (that is, A_1 is the set of rooms where all except the last room is red, A_2 is the set of colorings where the first $n-2$ rooms are red and the second to last is not red and the set A_n is the set of colorings of rooms where the first room is not red). We have broken up the set of colorings into n disjoint sets and $A = \bigcup_{k=1}^{n} A_k$. Moreover since A_k consists of $n-k-2$ red rooms, followed by a non-red room (in $x-1$ ways of coloring that room), followed by a coloring of the remaining $k-1$ rooms with x

possible colors, then $|A_k| = (x-1)x^{k-1}$ by the multiplication principle. Therefore

$$x^n - 1 = |A| = \sum_{k=1}^{n} |A_k| = \sum_{k=1}^{n} (x-1)x^{k-1} = (x-1) + (x-1)x + (x-1)x^2 + \cdots + (x-1)x^{n-1} .$$

Why make this proof so complicated? (we could have done it with a bit of algebra in 1/10th the time) What is worse that we have only managed to prove this identity for integers $x, n \geq 1$. It is hard to justify this sort of mathematical wank for this particular formula. It does give us a means of practicing when we are able to prove much more complicated combinatorial arguments. For example, to prove the Chu-Vandermonde formula

$$\binom{a+b+c+1}{a} = \sum_{k=0}^{a} \binom{k+b}{b} \binom{a-k+c}{c}$$

combinatorics is the easiest and it can even be used to make sure that your answer is right (see p. 50-51 in your book).

I should remark that even though we have given a proof of this identity for non-negative integers, we know that it is true for all integer values $n \geq 1$ and real numbers x. The reason is if we fix $n \geq 1$ then we have shown for all non-negative integer $x \geq 1$,

$$x^n - 1 - ((x-1) + (x-1)x + \cdots + (x-1)x^{n-1}) = 0.$$

But a polynomial of degree n has at most n zero points so if you tell me that $x^n - 1 - ((x-1) + (x-1)x + \cdots + (x-1)x^{n-1})$ is 0 for all positive integers, the only way that can happen is if it is the 0 polynomial and evaluates to 0 for all real values x.

For this argument I am using the following **FACT** which I have not shown you:
If $p(x)$ is a polynomial of degree $n > 0$, then there are at most n different real values x_i s.t. $p(x_i) = 0$.

I want to start in on sequences and so I gave a short introduction before we had to finish. Consider the following sequences

$$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{4}{0}, \dots \qquad 1, 1, 1, 1, 1, 1, 1, \dots$$
$$\binom{0}{1}, \binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \dots \qquad 0, 1, 2, 3, 4, 5, 6 \dots$$
$$\binom{0}{2}, \binom{1}{2}, \binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \dots \qquad 0, 0, 1, 3, 6, 10, 15, \dots$$
$$\binom{0}{3}, \binom{1}{3}, \binom{2}{3}, \binom{3}{3}, \binom{4}{3}, \dots \qquad 0, 0, 0, 1, 4, 10, 20, \dots$$
$$\binom{0}{4}, \binom{1}{4}, \binom{2}{4}, \binom{3}{4}, \binom{4}{4}, \dots \qquad 0, 0, 0, 0, 1, 5, 15 \dots$$

We have kind of written the table of binomial coefficients but we have selected the columns as our sequences to look at (we will also consider rows and diagonals too).

For any sequence of numbers $a_0, a_1, a_2, a_3, \dots$ (if you have a finite sequence of numbers then put 0's at the end), we define the *generating function* of the sequence as the series

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \ .$$

Remark 1: The first mistake that a lot of people make about the generating function and the sequence is that they are not the same thing. For every sequence we have a generating function and for every generating function we can come up with a sequence. They are not equal. We use the phrases '...the generating function for/of a sequence...' and '...the sequence whose generating function is...' but please don't mix the two things up.

Remark 2: A generating function neither generates, nor (at least in our case) is it a function (although it looks like one). x is an indeterminate. x does not have a value. x is a placeholder. Sometimes I will use other variables instead of x, but those will also be unknowns and we are working in a space where it makes sense to manipulate the variable algebraically.

We all know the geometric series:

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

(remember $(1-x)\cdot(1+x+x^2+x^3+x^4+\cdots) = 1+x+x^2+x^3+x^4+\cdots-x-x^2-x^3-x^4- = 1$ so divide by $(1-x)$).

Notice that the second sequence has a generating function

$$x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + \cdots$$

If you want a formula for it differentiate the first sequence to get $1+2x+3x^2+4x^3+5x^4+6x^5+\cdots$ and then multiply by x. That is,

$$x+2x^2+3x^3+4x^4+5x^5+6x^6+\cdots = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^2}\ .$$

NOTES ON SEPT 27, 2012

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We started to experiment a bit with generating functions and manipulate them and come up with formulas. We wrote down a bunch of sequences that we were able to give formulas for their generating functions. Recall that on Tuesday I had said look at the sequences

$$\binom{0}{k}, \binom{1}{k}, \binom{2}{k}, \binom{3}{k}, \binom{4}{k}, \ldots$$

If you look for $k = 1, 2, 3, \ldots$ then you can conjecture that there is a relatively simple formula for the generating function

$$\binom{0}{k} + \binom{1}{k} x \binom{2}{k} x^2 \binom{3}{k} x^3 \binom{4}{k} x^4 + \cdots = \sum_{n \geq 0} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} .$$

Proof. Take the derivative of $1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n \geq 0} x^n = \frac{1}{1-x}$. We have (by a quick induction argument), that

$$\frac{d^k}{dx^k} \frac{1}{1-x} = \frac{k!}{(1-x)^{k+1}}$$

We also know that

$$\frac{d^k}{dx^k} \frac{1}{1-x} = \frac{d^k}{dx^k} \sum_{n \geq 0} x^n = \sum_{n \geq 0} n(n-1)(n-2) \cdots (n-k+1) x^{n-k}$$

Therefore

$$\frac{1}{(1-x)^{k+1}} = \sum_{n \geq 0} \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} x^{n-k}$$

But the binomial coefficient $\binom{n}{k}$ is exactly the coefficient in this sum since

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)(n-k)(n-k-1) \cdots 2 \cdot 1}{k!(n-k)(n-k-1) \cdots 2 \cdot 1} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} .$$

Therefore

$$\frac{1}{(1-x)^{k+1}} = \sum_{n \geq 0} \binom{n}{k} x^{n-k}$$

and

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n \geq 0} \binom{n}{k} x^n . \quad \square$$

This corresponds to looking at columns of Pascal's triangle. We can also look at rows

$$\binom{1}{0}, \binom{1}{1}, \binom{1}{2}, \binom{1}{3}, \binom{1}{4}, \ldots = 1, 1, 0, 0, 0, 0, \ldots$$
$$\binom{2}{0}, \binom{2}{1}, \binom{2}{2}, \binom{2}{3}, \binom{2}{4}, \ldots = 1, 2, 1, 0, 0, 0, \ldots$$
$$\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}, \binom{3}{4}, \ldots = 1, 3, 3, 1, 0, 0, \ldots$$
$$\vdots$$
$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots$$

These have generating functions $(1+x)$, $(1+x)^2$, $(1+x)^3$ and the general sequence has generating function

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k \ .$$

Proof. left to the reader. easiest to do this by induction on n. □

Then I suggested we look at sequences like $1, 2, 3, 4, 5, \ldots$ and $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$ and $1^3, 2^3, 3^3, 4^3, 5^5, \ldots$. I looked at

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots \ = \sum_{n\geq 0}(n+1)x^n.$$

If you multiply by x and then take the derivative then you get the generating function for the squares because

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + \cdots = \sum_{n\geq 0}(n+1)x^{n+1}$$

and

$$\frac{1+x}{(1-x)^3} = \frac{d}{dx}\frac{x}{(1-x)^2} = 1 + 4x + 9x^2 + 16x^3 + 25x^4 + 36x^5 + \cdots = \sum_{n\geq 0}(n+1)^2x^n \ .$$

At this point I was using the computer at a regular basis. I went to the website `www.sagemath.org` and I had registered for an account. I used that account to do some of the calculations.

```
sage: taylor((1+x)/(1-x)^3,x,0,15)

256*x^15 + 225*x^14 + 196*x^13 + 169*x^12 + 144*x^11 + 121*x^10 + 100*x^9
+ 81*x^8 + 64*x^7 + 49*x^6 + 36*x^5 + 25*x^4 + 16*x^3 + 9*x^2 + 4*x + 1

sage: diff(x/(1-x)^2,x)
```

```
1/(x - 1)^2 - 2*x/(x - 1)^3

sage: factor(diff(x/(1-x)^2,x))

-(x + 1)/(x - 1)^3
```

The first command takes the taylor series of the expression $\frac{1+x}{(1-x)^3}$, the second command takes the derivative of $\frac{x}{(1-x)^2}$ and (since that wasn't presented as a single fraction) the third command factored the rational expression and showed it was equal to $-\frac{x+1}{(x-1)^3}$.

Then I said, what if I wanted to come up with a formula for the generating function $\sum_{n\geq 0}(n+1)^3x^n$? I should just multiply the last result by x and then differentiate. We find that

$$\frac{d}{dx}\left(x\frac{1+x}{(1-x)^3}\right) = \frac{d}{dx}\left(x\sum_{n\geq 0}(n+1)^2x^n\right) = \sum_{n\geq 0}(n+1)^3x^n$$

and I can use the computer to determine that:

```
sage: factor(diff(x*(1+x)/(1-x)^3,x))

(x^2 + 4*x + 1)/(x - 1)^4

sage: taylor((1+4*x+x^2)/(1-x)^4,x,0,14)

3375*x^14 + 2744*x^13 + 2197*x^12 + 1728*x^11 + 1331*x^10 + 1000*x^9 + 729*x^8
+ 512*x^7 + 343*x^6 + 216*x^5 + 125*x^4 + 64*x^3 + 27*x^2 + 8*x + 1
```

The last thing that I decided to do was look at what to do if we have a generating function for a sequence $a_0, a_1, a_2, a_3, \ldots$

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

and I want to know what the generating function was for the sequence of just the even terms $a_0, a_2, a_4, a_6, \ldots$. If I set $x \to -x$ then I see that

$$f(-x) = a_0 + a_1(-x) + a_2(-x)^2 + a_3(-x)^3 + a_4(-x)^4 + a_5(-x)^5 + a_6(-x)^6 + \cdots$$

then notice if we add $f(x) + f(-x)$ we have

$$f(x) + f(-x) = 2a_0 + 2a_2x^2 + 2a_4x^4 + 2a_6x^6 + \cdots$$

and then divide by 2

$$\frac{1}{2}(f(x) + f(-x)) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots$$

and then replace $x \to \sqrt{x}$, so that

$$\frac{1}{2}(f(\sqrt{x}) + f(-\sqrt{x})) = a_0 + a_2x + a_4x^2 + a_6x^3 + \cdots$$

and this is the generating function for the sequence $a_0, a_2, a_4, a_6, \ldots$.

I then did an example on the computer to convince us that it works as it should.

```
sage: f = (1+4*x+x^2)/(1-x)^4

sage: (f.subs(x=sqrt(x))+f.subs(x=-sqrt(x)))/2

1/2*(x + 4*sqrt(x) + 1)/(sqrt(x) - 1)^4 + 1/2*(x - 4*sqrt(x) + 1)/(sqrt(x) + 1)^4

sage: factor(_)

(x + 1)*(x^2 + 22*x + 1)/((sqrt(x) - 1)^4*(sqrt(x) + 1)^4)
```

(*) when I wrote `factor(_)` sage acted with the function `factor` on the last result the _ refers to the last result.

This says that the generating function for the odd cubes is given by

$$\sum_{n\geq 0}(2n+1)^3x^n = \frac{(1+x)(1+22x+x^2)}{(1-x)^4}$$

(note that if I was patient enough to do all the algebra on the blackboard I could have derived the same result by hand, but I don't have time to do all that in class).

If I want to check my answer, I find that

```
sage: taylor((1+x)*(1+22*x+x^2)/(1-x)^4,x,0,10)

9261*x^10 + 6859*x^9 + 4913*x^8 + 3375*x^7 + 2197*x^6 + 1331*x^5 + 729*x^4 +
343*x^3 + 125*x^2 + 27*x + 1
```

I suggested that for next time that you try to do the same thing except pick out every third term. What you need to do this is a little complex numbers. Everyone told me that this isn't common knowledge (as I assumed it should be). So here is a little summary:

$$i = \sqrt{-1}$$
$$e^{\theta i} = cos(\theta) + isin(\theta)$$

an r^{th} root of unity is given by the formula $\zeta_r = e^{2\pi i/r}$ because

$$(\zeta_r)^r = e^{2\pi i} = cos(2\pi) + isin(2\pi) = 1$$
$$1 + \zeta_r + \zeta_r^2 + \cdots + \zeta_r^{r-1} = 1\ .$$

What you want to do to generalize the formula for picking out every other term to every third term is to think of -1 as a second root of unity since $\zeta_2 = e^{\pi i} = -1$ and $1 + \zeta_2 = 0$ so instead of $f(x) + f(\zeta_2 x)$ you want something else.

NOTES ON OCT 2, 2012

MIKE ZABROCKI

I started off with an example that used complex numbers and this was not quite familiar to everyone. Last time we figured out that if we started with the generating function $A(x) = a_0 + a_1x + a_2x^2 + \cdots$, then it is possible to give the generating function for just the even terms in a three step process. First add $A(x)$ and $A(-x)$ and we find the generating function for $2a_0, 0, 2a_2, 0, 2a_4, 0, 2a_6, 0, \ldots$

$$A(x) + A(-x) = 2a_0 + 2a_2x^2 + 2a_4x^4 + 2a_6x^6 + \cdots$$

then divide by two and find the generating function for $a_0, 0, a_2, 0, a_4, 0, a_6, 0, \ldots$,

$$\frac{1}{2}(A(x) + A(-x)) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots$$

then replace x with $\sqrt{x}$ and find

$$\frac{1}{2}(A(\sqrt{x}) + A(-\sqrt{x})) = a_0 + a_2x + a_4x^2 + a_6x^3 + \cdots$$

and this is the generating function for the sequence $a_0, a_2, a_4, a_6, \ldots$.

Now what if we wanted to generalize this process to pick out every third term instead of every second? For this we need to know why every other term of the sequence cancelled. The reason is that $1^r + (-1)^r = 0$ if r is odd, and $1^r + (-1)^r = 2$ if r is even. The generalization of this statement is in complex numbers.

$$e^{ix} = cos(x) + isin(x)$$

If I set $\zeta_r = e^{2\pi i/r}$ (this is a definition), then $(\zeta_r)^r = e^{2\pi i} = 1$ and so

$$0 = (\zeta_r)^r - 1 = (\zeta_r - 1)(\zeta_r^{r-1} + \zeta_r^{r-2} + \cdots + \zeta_r + 1)$$

now since $\zeta_r - 1$ is not 0 and the product is 0, this means that $\zeta_r^{r-1} + \zeta_r^{r-2} + \cdots + \zeta_r + 1 = 0$.

Example: $\zeta_3 = e^{2\pi i/3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\zeta_3^2 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2})^2 = \frac{1}{4} - \frac{3}{4}) - i\frac{\sqrt{3}}{2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Then we see

$$\zeta_3 + \zeta_3^2 + 1 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2}) + (-\frac{1}{2} - i\frac{\sqrt{3}}{2}) + 1 = 0\ .$$

Example: $\zeta_2 = -1$, and $\zeta_2 + 1 = 0$.

Example: $\zeta_4 = I$, and $\zeta_4^2 = -1$, $\zeta_4^3 = -I$ and so

$$1 + \zeta_4 + \zeta_4^2 + \zeta_4^3 = 1 + I - 1 - I = 0 \ .$$

This is what we use to generalize what we did for the $r = 2$ case to pick out every other term. Step 1 is to add up $A(x)$, $A(\zeta_3 x)$ and $A(\zeta_3^2 x)$. We see

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$A(\zeta_3 x) = a_0 + a_1 \zeta_3 x + a_2 \zeta_3^2 x^2 + a_3 x^3 + a_4 \zeta_3 x^4 + a_5 \zeta_3^2 x^5 + a_6 x^6 + \cdots$$

$$A(\zeta_3^2 x) = a_0 + a_1 \zeta_3^2 x + a_2 \zeta_3 x^2 + a_3 x^3 + a_4 \zeta_3^2 x^4 + a_5 \zeta_3 x^5 + a_6 x^6 + \cdots$$

and so their sum is equal to

$$A(x) + A(\zeta_3 x) + A(\zeta_3^2 x) = 3a_0 + a_1(1 + \zeta_3 + \zeta_3^2)x + a_2(1 + \zeta_3^2 + \zeta_3)x^2 + 3a_3 x^3 + a_4(1 + \zeta_3 + \zeta_3^2)x^4$$

$$+ a_5(1 + \zeta_3^2 + \zeta_3)x^5 + 3a_6 x^6 + \cdots = 3a_0 + 3a_3 x^3 + 3a_6 x^6 + \cdots$$

This is the generating function for $3a_0, 0, 0, 3a_3, 0, 0, 3a_6, 0, 0, \ldots$. The next step is to divide this expression by 3 and the final step is to replace x by $\sqrt[3]{x}$. The final result is

$$\frac{1}{3}(A(\sqrt[3]{x}) + A(\zeta_3 \sqrt[3]{x}) + A(\zeta_3^2 \sqrt[3]{x})) = a_0 + a_3 x + a_6 x^2 + \cdots$$

The example that I did in class worked OK on the computer, but I didn't know how to make the computer do the algebra for us. The suggestion was that we take every third term of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$. If we do this we should get the same expression back. We find that

```
sage: taylor(1/(1-x),x,0,10)
x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
sage: zeta3 = exp(2*pi*I/3); zeta3
sage: taylor(1/(1-x) + 1/(1-zeta3*x) + 1/(1-zeta3^2*x),x,0,10)
3*x^9 + 3*x^6 + 3*x^3 + 3
sage: taylor(1/3*(1/(1-x) + 1/(1-zeta3*x) + 1/(1-zeta3^2*x)).subs(x=x^(1/3)),x,0,10)
x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
```

So what this shows is that the series for this expression has the same series as `1/(1-x)` but I couldn't figure out how to make the package do the simplification and show that

$$\frac{1}{3}\left(\frac{1}{1 - \sqrt[3]{x}} + \frac{1}{1 - \zeta_3 \sqrt[3]{x}} + \frac{1}{1 - \zeta_3^2 \sqrt[3]{x}}\right) = \frac{1}{1 - x}$$

instead you have to do the algebra yourself....

$$\frac{1}{3}\left(\frac{1}{1-\sqrt[3]{x}}+\frac{1}{1-\zeta_3\sqrt[3]{x}}+\frac{1}{1-\zeta_3^2\sqrt[3]{x}}\right)=$$

$$\frac{1}{3}\left(\frac{(1-\zeta_3\sqrt[3]{x})(1-\zeta_3^2\sqrt[3]{x})+(1-\sqrt[3]{x})(1-\zeta_3^2\sqrt[3]{x})+(1-\sqrt[3]{x})(1-\zeta_3\sqrt[3]{x})}{(1-\sqrt[3]{x})(1-\zeta_3\sqrt[3]{x})(1-\zeta_3^2\sqrt[3]{x})}\right)=$$

$$\frac{1}{3}\left(\frac{(1-\zeta_3\sqrt[3]{x}-\zeta_3^2\sqrt[3]{x}+x^{2/3})+(1-\sqrt[3]{x}-\zeta_3^2\sqrt[3]{x}+\zeta_3^2x^{2/3})+(1-\sqrt[3]{x}-\zeta_3\sqrt[3]{x}+\zeta_3x^{2/3})}{(1-\sqrt[3]{x}-\zeta_3\sqrt[3]{x}+\zeta_3x^{2/3})(1-\zeta_3^2\sqrt[3]{x})}\right)=$$

$$\frac{1}{3}\left(\frac{3}{(1-\sqrt[3]{x}-\zeta_3\sqrt[3]{x}+\zeta_3x^{2/3}-\zeta_3^2\sqrt[3]{x}+\zeta_3^2x^{2/3}+x^{2/3}-x)}\right)=$$

$$\frac{1}{3}\left(\frac{3}{(1-x)}\right)=$$

$$\frac{1}{(1-x)}=$$

I recommend that you experiment both by hand and with the computer to see that complex numbers work the way that you think that they do. Since $x^2-y^2=(x+y)(x-y)$ then it is also the case that $x^2+y^2=(x+iy)(x-iy)$. So it is possible to divide one complex number of the form $a+bi$ by $c+di$ (where a,b,c,d are all real numbers) and you will be able to put it in the form $e+fi$ by multiplying by the appropriate thing to clear the denominator of the complex numbers. So as an exercise, I suggest you try to show that

$$\frac{a+bi}{c+di}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}\ .$$

I then jumped to simpler example. How do we shift the generating function for a sequence and multiply by coefficients, etc.

sequence	generating function	expression
$a_0,a_1,a_2,a_3,a_4,a_5,a_6,\ldots$	$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$	$A(x)$
$0,0,0,a_0,a_1,a_2,a_3,a_4,\ldots$	$a_0x^3+a_1x^4+a_2x^5+a_3x^6+a_4x^7+\cdots$	$x^3A(x)$
$a_3,a_4,a_5,a_6,a_7,a_8,a_9,\ldots$	$a_3+a_4x+a_5x^2+a_6x^3+a_7x^4+a_8x^5+\cdots$	$(A(x)-a_0-a_1x-a_2x^2)/x^3$
$0a_0,1a_1,2a_2,3a_3,4a_4,\ldots$	$a_1x+2a_2x^2+3a_3x^3+4a_4x^4+5a_5x^5+\cdots$	$xA'(x)$
$\binom{0}{k}a_0,\binom{1}{k}a_1,\binom{2}{k}a_2,\binom{3}{k}a_3,\ldots$	$\sum_{n\geq 0}\binom{n}{k}a_n$	$x^kA^{(k)}(x)$

I then showed how to get the generating function for the Fibonacci numbers. Define $F_0=1$ and $F_1=1$ and $F_{n+1}=F_n+F_{n-1}$ for $n\geq 1$. The first few terms of the sequence are

$$1,1,2,3,5,8,13,21,34,55,\ldots$$

By definition the generating function is given by

$$F(x) = \sum_{n\geq 0} F_n x^n = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots .$$

It follows then that,

$$\begin{aligned} F(x) &= 1 + x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n \\ &= 1 + (x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots) + (F_0 x^2 + F_1 x^3 + F_2 x^4 + F_3 x^5 + \cdots) \\ &= 1 + xF(x) + x^2 F(x) \end{aligned}$$

By rearranging the terms of this formula we have

$$F(x) - xF(x) - x^2 F(x) = (1 - x - x^2) F(x) = 1$$

so

$$F(x) = \frac{1}{1 - x - x^2} .$$

I quickly checked this on sage and found

```
sage: taylor(1/(1-x-x^2),x,0,10)
89*x^10 + 55*x^9 + 34*x^8 + 21*x^7 + 13*x^6 + 8*x^5 + 5*x^4 + 3*x^3
+ 2*x^2 + x + 1
```

I will use this next time to show formulas that relate the Fibonacci numbers.

Exercises: Find formulas for the following generating functions (you don't need to simplify the expressions, but use the tools that we have developed in the last few days to write down an an expression).

(1) $\sum_{n\geq 0} F_{3n} x^n$
(2) $\sum_{k\geq 0} \binom{n}{2k} x^{2k}$
(3) $\sum_{n\geq 0} \binom{2n+1}{3} x^n$
(4) $\sum_{n\geq 0} \binom{n}{3} x^{2n+1}$
(5) $\sum_{n\geq 0} \binom{n}{2} F_n x^n$
(6) $\sum_{n\geq 0} \binom{n}{2} F_{n+4} x^n$
(7) $\sum_{n\geq 0} \binom{n+2}{2} \binom{n-2}{2} x^n$

Given that $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$ are the generating functions for the sequences $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$ respectively, find an expression for the generating function for the following sequences.

(8) $a_0, 2a_1, 4a_2, 8a_3, 16a_4, \ldots$
(9) $0, a_1, 2^2 a_2, 3^2 a_3, 4^2 a_4, 5^2 a_5, \ldots$
(10) $a_0, a_0, a_1, a_1, a_2, a_2, a_3, a_3, \ldots$
(11) $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$
(12) $a_0, b_1, a_2, b_3, a_4, b_5, \ldots$

(13) $a_1, a_5, a_9, a_{13}, a_{15}, a_{19}, \ldots$
(14) $a_0 + b_0, a_0 - b_0, a_1 + b_1, a_1 - b_1, a_2 + b_2, a_2 - b_2, \ldots$

NOTES ON OCT 4, 2012

MIKE ZABROCKI

I had planned a midterm on Oct 11. I can't be there that day. I am canceling my office hours that day and I will be available on Tuesday Oct 9 from 4-5pm instead. I am tempted to give a take home miterm instead of the in class one (which is very limited by the time). We will see....

Consider what happens when you multiply two generating functions

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

and

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \cdots$$

then if you expand it term by term you see

$$f(x)g(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots$$

Observe that in the expansion that the coefficient of x^n (for $n = 0, 1, 2$ only because I didn't go further) that the subscripts of a_ib_j add up to the exponent of x. If we expand all terms of the series then we reason that this always happens and we have that the coefficient of x^n is $\sum_{i+j=n} a_ib_j$. That is,

$$f(x)g(x) = \sum_{n\geq 0} \left(\sum_{i+j=n} a_ib_j \right) x^n \ .$$

I declared that if a_r and b_s have a combinatorial meaning, then a_rb_s has a combinatorial meaning and so does $\sum_{r+s=n} a_rb_s$. I formulated this as a mathematical principle.

Principle 1. (**The Multiplication Principle of Generating Functions**) *Assume that a_r is equal to the number of widgets of 'size' r and b_s is equal to the number of doodles of 'size' s, then we say that $f(x)$ is the generating function for the number of widgets of 'size' n and $g(x)$ is the generating function for the number of doodles of 'size' n and*

$$f(x)g(x) = \sum_{n\geq 0} \left(\sum_{i+j=n} a_ib_j \right) x^n$$

is the generating function for the pairs of elements (x, y) where x is a widget of 'size' i and y is a doodle of 'size' j with $i + j = n$.

So what I have done is I have applied the addition principle and the multiplication principle to count such pairs (x, y) where x is a widget and y is a doodle where I break the set of pairs of 'size' n into those where x is of size i and y is of size $n - i$. In order to make the statement of the principle above I had to apply the addition principle so that the widget was of size i where $0 \leq i \leq n$.

Remark 2. *I intentionally put the word 'size' in quotes because I haven't been super precise about what I mean. This really means that if I group the objects that I am calling widgets into groups by a grading (something that happens often in combinatorics) then the word 'size' here represents an association with the grading. The word 'size' may not be accurate. Consider the example below when I am talking about change for n cents, then the 'size' in that case means the number of cents. I am using 'size' in an abstract way to mean whatever term you are grading by.*

Remark 3. *This notation of expressing $f(x)$ as the generating function for the number of widgets of 'size' n and $g(x)$ the generating function for the number of doodles of 'size' n is my own. You won't see it in the textbook and if you google the words 'widgets' and 'doodles' you are likely to find web pages written by me. I just find this a convenient way to think about combinatorics of generating functions in the case when the generating functions are for sequences of non-negative integers and there is a combinatorial interpretation for these integers. If $f(x)$ is the g.f. for widgets and $g(x)$ is the g.f. for doodles then $f(x)g(x)$ is this generating function for pairs consisting of a widget and a doodle (i.e. a widget-doodle).*

Let me give you an example of something we can apply this principle to. Consider the number of non-negative solutions to the equation $x_1 + x_2 = n$ for $n \geq 0$. If I write the generating function for the number of such solutions I can compute it in two different ways and get the same answer.

The first way is I will just look and notice that the non-negative solutions to the equation $x_1 + x_2 = n$ are $(x_1, x_2) \in \{(n, 0), (n-1, 1), (n-2, 2), \ldots, (0, n)\}$. Therefore the number of solutions to $x_1 + x_2 = n$ is equal to $n+1$ and the generating function $\sum_{n\geq 0}(n+1)x^n = \frac{1}{(1-x)^2}$.

Now let me try to compute the same thing using the multiplication principle of generating functions (MPofGFs). The a solution to $x_1 + x_2 = n$ is isomorphic to a solution to a pair (x_1, x_2) whose sum is n. By MPofGFs we have that

$$\sum_{n\geq 0}(\#\text{pairs } (x_1, x_2) \text{ s.t. } x_1 + x_2 = n)x^n = \left(\sum_{n\geq 0}(\#\text{solutions to the equation } x_1 = n)x^n\right)^2$$

But the number of solutions to the equation $x_1 = n$ is equal to 1 for all $n \geq 0$ so

$$\sum_{n\geq 0}(\#\text{solutions to the equation } x_1 = n)x^n = \frac{1}{1-x}$$

and hence

$$\sum_{n\geq 0}(\#\text{pairs } (x_1, x_2) \text{ s.t. } x_1 + x_2 = n)x^n = \frac{1}{(1-x)^2} \ .$$

I know it seems a kind of trivial example, but we have shown that the number of solutions to $x_1 + x_2 = n$ has generating function equal to $1/(1-x)^2$ in two different ways. Lets try to expand this.

The generating function for the number of non-negative solutions to

$$x_1 + x_2 + x_3 + x_4 = n$$

is equal to the number of tuples (x_1, x_2, x_3, x_4) where $x_1 + x_2 + x_3 + x_4 = n$ which is equal to the number of pairs (X, Y) where X is a pair (x_1, x_2) with $x_1 + x_2 = i$ and Y is a pair (x_3, x_4) with $x_3 + x_4 = n - i$. By the MPofGFs we know that

$$\sum_{n\geq 0}(\#\text{pairs } (X,Y) \text{ s.t. } X \text{ is a solution to } x_1+x_2=i \text{ and } Y \text{ is a solution to } x_3+x_4=n-i)x^n$$

$$= \left(\sum_{n\geq 0}(\text{pairs } (x_1,x_2) \text{ s.t. } x_1+x_2=n)x^n\right)^2 = \left(\frac{1}{(1-x)^2}\right)^2 = \frac{1}{(1-x)^4}\ .$$

In general, we can apply the MPofGFs multiple times to show that

$$\sum_{n\geq 0}\#(\text{number of solutions to } x_1+x_2+\cdots+x_k=n)x^n = \frac{1}{(1-x)^k}.$$

The thing is that this is something that we have already discussed in this class

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0}\binom{n+k-1}{k-1}x^n$$

so the number of solutions to $x_1 + x_2 + \cdots + x_k = n$ is equal to $\binom{n+k-1}{k-1}$. We had discussed this before that the number of solutions is equal to the number of sequences of n dots $\bullet$ and $k-1$ bars $|$.

Here is an example of a problem that we can apply these ideas to: "How many ways are there of making change for 78 using pennies, nickels, dimes, and quarters." The answer is equivalent to the number of tuples (p, n, d, q) such that $p+5n+10d+25q = n$. If we apply MPofGFs, then this is the product of the generating functions for solutions to $p = N$, the solutions to $5n = N$, the solutions to $10d = N$, the solutions to $25q = N$ and these sequences have respective generating functions $\frac{1}{1-x}$, $\frac{1}{1-x^5}$, $\frac{1}{1-x^{10}}$ and $\frac{1}{1-x^{25}}$.

Therefore the generating function for the number of ways of making change for N cents with pennies, nickels, dimes and quarters is

$$C(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}\ .$$

If in particular I wanted the number of ways of making change for 78 cents I would go to the computer and ask:

```
sage: taylor(1/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)),x,0,78).coefficient(x^78)
121
sage: taylor(1/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)),x,0,10)
```

```
4*x^10 + 2*x^9 + 2*x^8 + 2*x^7 + 2*x^6 + 2*x^5 + x^4 + x^3 + x^2 + x + 1
```

I also calculated here the ways of making change for N cents for $0 \leq N \leq 10$ and I notice that the number ways of making change for 10 cents is $4 = \#\{$10 pennies; 1 nickel, 5 pennies; 2 nickels; one dime $\}$ and this agrees with the answer that the generating function returns.

I then wanted to demonstrate that you can throw in some pretty crazy conditions on your combinatorial problem and calculating the number of such solutions is still a matter of breaking up the problem into pieces where you can either add or multiply generating functions. As long as your combinatorial condition has a nice expression for the generating function, then applying this tool works really well.

So, for instance say that in addition that you wanted to make change for N cents where you also have an American quarter and two American nickels but as many Canadian pennies, nickels, dimes and quarters as you want. You can break the combinatorial problem into the number of tuples (X, Y, Z) where X is some way of taking change for I cents with Canadian coins, Y is some way of taking change for J cents using the American quarter or not, Z is some way of making change for K cents using the two American nickels. We want to know how may ways there are of making change for N cents, so we will take the coefficient of x^N in the expression for the product of generating functions.

We already know that the generating function for the first part of the tuple is $C(x)$ (given above). With the American quarter we can make change either for 0 cents or 25 cents and only in one way each so the generating function is $1 + x^{25}$. With the two American nickels we can make change for $0, 5$ or 10 cents only and there is exactly one way of doing that (the nickels are indistinguishable), then the generating function is equal to $1 + x^5 + x^{10}$. Therefore the generating function for N cents where you also have an American quarter and two American nickels but as many Canadian pennies, nickels, dimes and quarters as you want is equal to

$$C(x)(1 + x^{25})(1 + x^5 + x^{10}) = \frac{(1 + x^{25})(1 + x^5 + x^{10})}{(1 - x)(1 - x^5)(1 - x^{10})(1 - x^{25})} .$$

We can compute the number of these by asking the computer:

```
sage: taylor((1+x^5+x^10)*(1+x^25)/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)),
        x,0,78).coefficient(x^78)
430
```

We can also use generating function to derive combinatorial identites. Recall that last time, I showed that the generating function for the Fibonacci numbers is $1/(1 - x - x^2) = F(x) = \sum_{n \geq 0} F_n x^n$. Then we can rewrite this as

$$F(x) = \frac{1}{1 - (x + x^2)} = \sum_{n \geq 0} (x + x^2)^n = \sum_{n \geq 0} (1 + x)^n x^n$$

We also know that $(1+x)^n$ is the generating function for the binomial coefficients $(1+x)^n = \sum_{k\geq 0}\binom{n}{k}x^k$ therefore

$$F(x) = \sum_{n\geq 0}\sum_{k\geq 0}\binom{n}{k}x^{n+k}$$

If I take the coefficient of x^m in both sides of this equation I find that

$$F_m = \sum_{n+k=m}\binom{n}{k} .$$

For example

$$F_5 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} + \binom{2}{3} + \binom{1}{4} + \binom{0}{5} .$$

I know that $\binom{2}{3}+\binom{1}{4}+\binom{0}{5} = 0$ and $\binom{5}{0} = 1$, $\binom{4}{1} = 4$ and $\binom{3}{2} = 3$, therefore $F_5 = 1+4+3 = 8$.

$$F_6 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13$$

and this agrees with our generating function

```
sage: taylor(1/(1-x-x^2),x,0,8)
34*x^8 + 21*x^7 + 13*x^6 + 8*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
```

We can also derive a second equation for the Fibonacci numbers. If you apply the quadratic formula to $1 - x - x^2 = 0$ you obtain that $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{1-\sqrt{5}}{2}$ are the roots of the equation. Check explicitly that $\phi\overline{\phi} = -1$ and $\phi + \overline{\phi} = 1$, therefore

$$(1 - \phi x)(1 - \overline{\phi}x) = 1 - \phi x - \overline{\phi}x + \phi\overline{\phi}x^2 = 1 - x - x^2$$

Now if I have a rational function of the form $\frac{1}{(1-\phi x)(1-\overline{\phi}x)}$ then there is this technique that you probably learned in calculus that says that there exists A and B such that

$$F(x) = \frac{1}{(1 - \phi x)(1 - \overline{\phi}x)} = \frac{A}{(1 - \phi x)} + \frac{B}{(1 - \overline{\phi}x)} .$$

If we take the coefficient of x^m in both sides of this equation we find that

$$F_m = A\phi^m + B\overline{\phi}^m .$$

If you solve for A and B by saying that since $A(1 - \overline{\phi}x) + B(1 - \phi x) = 1$, then let $x = 1/\overline{\phi}$ to see that $B = \frac{1}{1-\phi/\overline{\phi}} = \frac{\overline{\phi}}{\overline{\phi}-\phi} = -\frac{\overline{\phi}}{\sqrt{5}}$ and let $x = 1/\phi$ so then $A = \frac{\phi}{\phi-\overline{\phi}} = \frac{\phi}{\sqrt{5}}$. We conclude

$$F_m = \frac{\phi^{m+1}}{\sqrt{5}} - \frac{\overline{\phi}^{m+1}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{m+1} .$$

which (at least to me) is kind of hard to believe until you do this by hand or test it out on the computer.

```
sage: expand( 1/sqrt(5)*(((1+sqrt(5))/2)^6-((1-sqrt(5))/2)^6))
8
sage: expand( 1/sqrt(5)*(((1+sqrt(5))/2)^7-((1-sqrt(5))/2)^7))
13
```

Exercises: I was asked about what would be questions at the level of a test question. You should be able to answer

(a) during a timed exam,
(b) during a timed exam (this should be at a slightly harder level maybe you will need a computer to get the numerical value)
(c) would be at the level of a homework problem or a take home exam
(d) should be considered a challenge and (while doable) may take a while to complete

On the following two questions find a generating function representing the sequence for a all n. Take the coefficient of x^n for the n specified in the problem

(1) How many ways are there making change for $n = \$1.00$ with pennies, nickels, dimes and quarters such that:
  (a) there are an even number of nickels and no pennies ?
  (b) such that there at most 6 nickels ?
  (c) the total number of nickels and dimes is even ?
  (d) the total number of pennies, dimes and quarters is even ?
(2) How many ways are there of placing $n = 50$ balls in 10 distinguished boxes such that:
  (a) there is no restriction ?
  (b) there are at most 17 balls in the first box ?
  (c) the first 4 boxes have at most 10 of the balls ?
  (d) the first 4 boxes have at least half of the balls ?

(3) (a) Find the generating function for the sequence $a_0, 2a_1, a_2, 2a_3, a_4, 2a_5, a_6, 2a_7, \ldots$ in terms of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.
  (b) Find the generating function for the sequence $a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6, \ldots$ in terms of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.
  (c) On the homework assignment you were to arrive at an expresion for $L(x) = \sum_{n\geq 0} L_n x^n = (1 + 2x)/(1 - x - x^2)$. Using the formula for the product of generating functions, what is the coefficient of x^n in the generating function $\frac{1}{1+2x}L(x)$? Conclude a formula relating the Fibonacci numbers and the Lucas numbers because $F(x) = \frac{1}{1+2x}L(x)$.
  (d) Given $D_0 = 1$, $D_1 = a$ and $D_{n+1} = aD_n + bD_{n-1}$ where a, b are unknowns. The entry sequence D_n will be a polynomial in a and b. Find the coefficient of $a^r b^s$.

Enumeration problems (this is not related to generating functions, it is a review of the types of combinatorial problems that came up on the last homework): Make sure that you explain your answer as completely as possible. It is not sufficient to give just a numerical answer, you must give an explanation why your answer is correct.

(4) How many 5 card hands ...
   (a) contain a three of a kind and a 3 values in a row ?
   (b) contain a three of a kind sequence and 3 values in a row that are not all of the same suit ?
   (c) contain a three of a kind sequence and 3 values in a row that are not all of the same suit but do not contain a Queen?
   (d) contain a three of a kind sequence and 3 values in a row that are not all of the same suit but do not contain a Queen or a black 10?

   Note: the three of a kind and the three value sequence must overlap $4\heartsuit 5\diamondsuit 5\clubsuit 5\spadesuit 6\clubsuit$.

NOTES ON OCT 9, 2012

MIKE ZABROCKI

Exercises: I said that I would solve problems that people asked me about in class. I am going to put the solutions to these (the ones that people asked about) so that you have a reasonable idea of my expectations of what I would like to see as justification of these problems.

(7) $\sum_{n\geq 0} \binom{n+2}{2}\binom{n-2}{2} x^n$

(1) How many ways are there making change for $n = \$1.00$ with pennies, nickels, dimes and quarters such that:
  (a) there are an even number of nickels and no pennies ?
  (b) such that there at most 6 nickels ?
  (c) the total number of nickels and dimes is even ?
  (d) the total number of pennies, dimes and quarters is even ? (*)

(2) How many ways are there of placing $n = 50$ balls in 10 distinguished boxes such that:
  (c) the first 4 boxes have at most 10 of the balls ?
  (d) the first 4 boxes have at least half of the balls ? (*)

(3) (b) Find the generating function for the sequence $a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6, \ldots$ in terms of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.
  (d) Given $D_0 = 1$, $D_1 = a$ and $D_{n+1} = aD_n + bD_{n-1}$ where a, b are unknowns. The entry sequence D_n will be a polynomial in a and b. Find the coefficient of $a^r b^s$.

(*) I will not be able to do (1) (d) and (2) (d) here. I will post a solution at a later date.

(7) I want to give an expression for $\sum_{n\geq 0} \binom{n+2}{2}\binom{n-2}{2} x^n$. I will use one fact that we derived on Sept 27 notes, that if $A(x) = \sum_{n\geq 0} a_n x^n$, then $\frac{x^k}{k!} A^{(k)}(x) = \sum_{n\geq 0} \binom{n}{k} a_n x^n$. I will also use the fact that $x^r A(x) = \sum_{n\geq 0} a_n x^{n+r} = \sum_{n\geq r} a_{n-r} x^n$. Start with $A(x) = \frac{1}{(1-x)^3} = \sum_{n\geq 0} \binom{n+2}{2} x^n$. Notice that since $\binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$ then what I would like to do is decrease the exponent of the x in $\binom{n+2}{2} x^n$ in $A(x)$ by 2 (to $n-2$) then differentiate twice and then multiply until we have the right exponent.

$$A(x) - 1 - 3x = \sum_{n \geq 2} \binom{n+2}{2} x^n$$

$$x^{-2}(A(x) - 1 - 3x) = \sum_{n \geq 2} \binom{n+2}{2} x^{n-2}$$

$$\frac{d^2}{dx^2}\left(x^{-2}(A(x) - 1 - 3x)\right) = \sum_{n \geq 2} \binom{n+2}{2} \frac{d^2}{dx^2}(x^{n-2}) = \sum_{n \geq 2} \binom{n+2}{2} (n-2)(n-3)x^{n-4}$$

$$\frac{1}{2}\frac{d^2}{dx^2}\left(x^{-2}(A(x) - 1 - 3x)\right) = \sum_{n \geq 2} \binom{n+2}{2} \frac{(n-2)(n-3)}{2} x^{n-4}$$

$$x^4 \frac{1}{2}\frac{d^2}{dx^2}\left(x^{-2}(A(x) - 1 - 3x)\right) = \sum_{n \geq 2} \binom{n+2}{2}\binom{n-2}{2} x^n = \sum_{n \geq 0} \binom{n+2}{2}\binom{n-2}{2} x^n$$

(1) (a) Every way of making change for n cents using an even number of nickels and some dimes and quarters is a tuple of the form (X, Y, Z) where X is some means of making change for r cents with an even number of nickels, Y is some means of making change for s cents using dimes and Z is some means of making change for $n - r - s$ cents with quarters. Therefore we have that the generating function for making change with an even number of nickels, and some dimes and quarters is equal to

= (g.f. for the ways of making change using an even number of nickels)(g.f. for the ways of making change using dimes)(g.f. for the ways of making change using quarters)

If I am using an even number of nickels, then I can make change for n cents in one way if and only if n is a multiple of 10. Therefore

$$\text{g.f. for the ways of making change using an even number of nickels} = \frac{1}{1-x^{10}}$$

Similarly, if I am making change for n cents using dimes , then I can make change for n cents in one way if and only if n is a multiple of 10. Therefore

$$\text{g.f. for the ways of making change using dimes} = \frac{1}{1-x^{10}}$$

If I am making change for n cents using quarters, then I can make change for n cents in one way if and only if n is a multiple of 25.

$$\text{g.f. for the ways of making change using quarters} = \frac{1}{1-x^{25}}$$

We conclude that the g.f. for the number of ways of making change for n cents using an even number of nickels and some dimes and quarters is

$$\frac{1}{(1-x^{10})^2(1-x^{25})}$$

Ideally we would like to take the coefficient of x^{100} and in this case the generating function is simple enough that we can do this by hand. Since $\frac{1}{1-x^{25}} = 1+x^{25}+x^{50}+x^{75}+x^{100}+\cdots$ then

$$\frac{1}{(1-x^{10})^2(1-x^{25})}\Big|_{x^{100}} = \frac{1}{(1-x^{10})^2}\Big|_{x^{100}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{75}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{50}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{25}} + \frac{1}{(1-x^{10})^2}\Big|_{x^{0}}$$

Well we know that there is no way of getting a power of x^{25} or x^{75} in $1/(1-x^{10})^2$ because the only powers that appear are multiples of 10. Moreover we also know that $1/(1-x^{10})^2 = 1+2x^{10}+3x^{20}+4x^{30}+5x^{40}+6x^{50}+\cdots$. Therefore,

$$\frac{1}{(1-x^{10})^2(1-x^{25})}\Big|_{x^{100}} = 11+6+1 = 18 \ .$$

(1) (b) Every way of making change with pennies, dimes, quarters and 6 nickels can be broken down into four steps consisting of a way of making change in pennies for r cents, a way of making change in dimes for s cents, a way of making change with quarters for t cents and with at most 6 nickels for $n-r-s-t$ cents. By the multiplication principle of generating functions, we know that

g.f. for making change for n cents with pennies, dimes, quarters and at most 6 nickels

= (g.f. for making change for n cents with pennies)(the ways of making change using dimes)(g.f. for the ways of making change using quarters)(the ways of making change using at most 6 nickels)

The generating functions for making change with pennies, dimes and quarters are similar to the last problem and are respectively $\frac{1}{1-x}$, $\frac{1}{1-x^{10}}$, $\frac{1}{1-x^{25}}$. The g.f. for making change using at most 6 nickels is slightly different. Then there is exactly one way of making change for $0, 5, 10, 15, 20, 25, 30$ cents using at most 6 nickels hence the generating function is given by

$$1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+x^{30} = \frac{1-x^{35}}{1-x^5}$$

Therefore

g.f. for making change for n cents with pennies, dimes, quarters and at most 6 nickels

$$= \frac{1-x^{35}}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

(1) (c) The number of ways of making change for n cents with pennies, quarters and then an even number of nickels and dimes can be seen a way of making change for r cents using pennies, followed by s cents using quarters, followed by $n-r-s$ cents using an even number of nickels and dimes. Therefore,

g.f. for making change for n cents using pennies, quarters and then an even number of nickels and dimes = (g.f. for making change for n cents using pennies)(g.f. for making change for n cents using quarters)(g.f. for making change for n cents using even number of nickels and dimes)

Now there is one way of making change for n cents using pennies hence the generating function is $\frac{1}{1-x}$.

There is one way of making change for n cents using quarters if and only if n is divisible by 25, hence the generating function for making change for n cents using quarters is $\frac{1}{1-x^{25}}$.

Now to make change for n cents with an even number of nickels and dimes is only possible if n is divisible by 5. In fact, if n is divisible by 20 then there is one way to make change for 0 cents and one more way for each 20 cents more and so the generating function for these terms is $1/(1-x^{20}) = 1 + 2x^{20} + 3x^{40} + 4x^{60} + \cdots$. If $n \equiv 5$ $(mod\ 20)$, then there are no ways of making change for 5 cents with an even number of coins and for every 20 cents there is one more way so the generating function for these terms is $x^{25}/(1-x^{20})^2 = x^{25} + 2x^{45} + 3x^{65} + 4x^{85} + \cdots$. If $n \equiv 10$ $(mod\ 20)$ then there is one way for making change for 10 cents (2 nickels) and one more way for each 20 cents, so the generating function for these terms is $x^{10}/(1-x^{20})^2 = x^{10} + 2x^{30} + 3x^{50} + 4x^{70} + \cdots$. If $n \equiv 15$ $(mod\ 20)$ there is one way of making change for 15 cents (one nickel and one dime) and one more way for each 20 cents after so the generating function for these terms is $x^{15}/(1-x^{20})^2 = x^{15} + 2x^{35} + 3x^{55} + 4x^{75} + \cdots$. Since every multiple of 5 is equivalent to 0, 5, 10 or 15 $(mod\ 20)$ then the generating function for the number of ways of making change for n cents using an even number of nickels and dimes is equal to the sum of the generating functions for n equivalent to 0, 5, 10, or 15 $(mod\ 20)$, therefore the generating function is equal to $(1 + x^{10} + x^{15} + x^{25})/(1-x^{20})^2$.

We conclude that

g.f. for making change for n cents using pennies, quarters and then an even number

$$\text{of nickels and dimes} = \frac{1 + x^{10} + x^{15} + x^{25}}{(1-x)(1-x^{25})(1-x^{20})^2}$$

(3) (b) We know from class Sept 27 that if $A(x) = \sum_{n\geq 0} a_n x^n$, then

$$x\frac{1}{2}\left(A(x) + A(-x)\right) = a_0 x + a_2 x^3 + a_4 x^5 + a_6 x^7 + \cdots$$

and the generating function for the odd terms (shifted) is

$$\frac{1}{2x}\left(A(x) - A(-x)\right) = a_1 + a_3 x^2 + a_5 x^4 + a_7 x^6 + \cdots$$

hence

$$\frac{x}{2}\left(A(x) + A(-x)\right) + \frac{1}{2x}\left(A(x) - A(-x)\right) = a_1 + a_0 x + a_3 x^2 + a_2 x^3 + a_5 x^4 + a_4 x^5 + a_7 x^6 + a_6 x^7 + \cdots$$

(3)(d) Since $D_0 = 1$ and $D_1 = a$, then $D(x) = \sum_{n\geq 0} D_n x^n = 1 + ax + \sum_{n\geq 2} D_n x^n$, now use the recursive definition for $n \geq 2$, so that

$$D(x) = 1 + ax + \sum_{n\geq 2}(aD_{n-1} + bD_{n-2})x^n = 1 + axD(x) + bx^2 D(x)$$

Solving for $D(x)$, we have $D(x) - axD(x) - bx^2 D(x) = D(x)(1 - ax - bx^2) = 1$, so then

$$D(x) = \frac{1}{1 - ax - bx^2}\ .$$

Now you are asked what is the coefficient of $a^r b^s$ in the coefficient of x^n. To do this we expand as a series in x, then look at the resulting expression and take the coefficient of $a^r b^s$.

$$\frac{1}{1 - ax - bx^2} = \frac{1}{1 - (ax + bx^2)} = \sum_{m\geq 0}(ax+bx^2)^m = \sum_{m\geq 0}(a+bx)^m x^m = \sum_{m\geq 0}\sum_{k\geq 0}\binom{m}{k} a^{m-k} b^k x^{m+k}\ .$$

Now the coefficient of x^n forces $n = m + k$, hence

$$D_n = \sum_{k\geq 0}\binom{n-k}{k} a^{n-2k} b^k.$$

The coefficient of $a^r b^s$ is equal to 0 unless $k = s$ and $n - 2k = r$ (or $n = r + 2s$) and if $n = r + 2s$ then the coefficient is equal to $\binom{r+s}{s}$.

NOTES ON OCT 16, 2012

MIKE ZABROCKI

I started off by giving an example that was typical of the type of problem that I have been giving in the homework and the midterm. I felt that at this point you should be prepared to this type of problem:

How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = n$$

with $x_1 + x_2$ divisible by 3?

The first step would be to find an equation for the generating function, although there is a second answer (that I didn't discuss in class) which also can be used to answer this question. To find the generating function you would first break the problem into three steps, find the number of solutions to $x_1 + x_2 = k$ where $x_1 + x_2$ is divisible by 3, the number of solutions to $x_3 = \ell$ and the number of solutions to $x_4 = n - k - \ell$. Since we have broken down the problem into these three substeps, then we know that

g.f. for # of solutions to $x_1 + x_2 + x_3 + x_4 = n$ with $x_1 + x_2$ divisible by 3 = (g.f. for # of solutions to $x_1 + x_2 = n$ with $x_1 + x_2$ divisible by 3) (g.f. for # of solutions to $x_3 = n$) (g.f. for # of solutions to $x_4 = n$). We know that for each n there is one solution to $x_3 = n$ so the generating function for the number of such solutions is $1/(1 - x)$ (similarly for $x_4 = n$). Now to find the generating function for $x_1 + x_2 = n$ with $x_1 + x_2$ divisible by 3, the obvious way is to take the generating function for the number of solutions to $x_1 + x_2 = n$ which we know is equal to $A(x) = 1/(1 - x)^2$ and then pick out every third term using the method that we discussed on October 2 (see notes) and give it was

$$\frac{1}{3}(A(x) + A(\zeta_3 x) + A(\zeta_3^2 x))$$

Instead I will suggest another method to get at this generating function by computing a table of coefficients and then writing a formula for the generating function.

n	0	1	2	3	4	5	6	7	8	9
# of solutions	1	0	0	4	0	0	7	0	0	10

In other words if n is divisible by 3, then the number of solutions is $n + 1$, otherwise it is 0. This means that the generating function is

$$\sum_{n\geq 0}(3n + 1)x^{3n} = 3\sum_{n\geq 0} nx^{3n} + \sum_{n\geq 0} x^{3n} = 3\frac{x^3}{(1 - x^3)^2} + \frac{1}{1 - x^3} .$$

Note that the last equality comes from the tables of generating functions that we have developed. From this we conclude that

g.f. for # of solutions to $x_1 + x_2 + x_3 + x_4 = n$ with $x_1 + x_2$ divisible by 3 =

$$\left(3\frac{x^3}{(1-x^3)^2} + \frac{1}{1-x^3}\right)\frac{1}{(1-x)^2} \ .$$

There is another way of coming up with an answer to this question. If we want to find the number of solutions to $x_1 + x_2 + x_3 + x_4 = n$ with $x_1 + x_2$ divisible by 3 then we don't need to go so far as to apply generating functions. This is equal to the number of solutions to $x_1 + x_2 = k$ and $x_3 + x_4 = n - k$ with $x_1 + x_2$ divisible by 3. For each k there are $(k + 1)$ solutions to $x_1 + x_2 = k$ and there are $n - k + 1$ solutions to $x_3 + x_4 = n - k$. Therefore the number of solutions

$$\sum_{3 \ divides \ k} (k+1)(n-k+1) = \sum_{r=0}^{\lfloor n/3 \rfloor} (3r+1)(n-3r+1).$$

What I hoped to show from this example is that ordinary generating functions are a very powerful tool for enumerating certain types of sets. Usually these are sets that can be reduced to something that is very similar to the example we just looked at. We can apply the multiplication principle of generating functions if we can divide the enumeration of a set with c_n elements into a widget of size k and a doodle of size $n - k$, then if a_k is the number of widgets of size k and b_{n-k} is the number of doodles of size $n - k$, then

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \ . \tag{1}$$

The problem is that there are many other enumeration questions where we don't have this sort of decomposition. One of those examples are the Bell numbers: $B_0 = 1$, $B_1 = 1$ and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ for $n > 1$ which is equal to the number of set partitions of $n + 1$. We can calculate the next few values as $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$.

The problem is that the expression $\sum_{k=0}^{n} \binom{n}{k} B_k$ is not of the form $\sum_{k=0}^{n} a_k b_{n-k}$. Why? If I set $B(x) = \sum_{n\geq 0} B_n x^n$, then when I multiply $B(x)A(x)|_{x^n}$ is $\sum_{k=0}^{n} a_{n-k} B_k$ and I can't find a generating function where $a_{n-k} = \binom{n}{k}$. It just doesn't seem to work.

There is a way around this. We can define a new type of generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and if we take a second to $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ and multiply these together then we see that

$$A(x)B(x) = \sum_{n\geq 0} \frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!}x^n = \sum_{n\geq 0} \frac{n!}{k!(n-k)!} a_k b_{n-k} \frac{x^n}{n!} = \sum_{n\geq 0} \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!}$$

.

This gives us a new principle to work with.

Principle 1. *The coefficient of $x^n/n!$ in the product of $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ is equal to*

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}. \tag{2}$$

I mention this because in the recurrence for B_{n+1} if we set $a_k = B_k$ and $b_{n-k} = 1$ then it is of this form. Therefore it seems as though we might be able to write down a generating function of this form. We call $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ the exponential generating function for a sequence. Consider the exponential generating function for the sequence $1, 1, 1, 1, 1, 1, \ldots,$

$$\sum_{n\geq 0} 1 \frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^n}{n!} = e^x \ .$$

The exponential generating function for the sequence $0, 1, 2, 3, 4, 5, 6, \ldots,$ is equal to

$$\sum_{n\geq 0} n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{(n-1)!} = xe^x \ .$$

Now consider the sequence $\binom{0}{k}, \binom{1}{k}, \binom{2}{k}, \binom{3}{k}, \binom{4}{k}, \binom{5}{k}, \ldots,$ where k is fixed. We calculate that the exponential generating function is equal to

$$\sum_{n\geq 0} \binom{n}{k} \frac{x^n}{n!} = \sum_{n\geq k} \frac{n!}{k!(n-k)!} \frac{x^n}{n!} = \sum_{n\geq k} \frac{1}{k!} \frac{x^n}{(n-k)!} = \frac{x^k}{k!} \sum_{n\geq k} \frac{x^{n-k}}{(n-k)!} = \frac{x^k}{k!} e^x \ .$$

Now lets apply what we know to finding a formula for the exponential generating function for $B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$ where $B_0 = B_1 = 1$ and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Lets work it out as we normally do except with exponential generating functions.

$$\begin{aligned} B(x) &= \sum_{n\geq 0} B_n \frac{x^n}{n!} \\ &= 1 + \sum_{n\geq 1} B_n \frac{x^n}{n!} \\ &= 1 + \sum_{n\geq 1} \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \frac{x^n}{n!} \\ &= 1 + B_0 \frac{x}{1!} + \left(\binom{1}{0} B_0 + \binom{1}{1} B_1 \right) \frac{x^2}{2!} + \left(\binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 \right) \frac{x^3}{3!} + \cdots \end{aligned}$$

Now those coefficients that are appearing in this sum should look very familiar. They are exactly those that appear in equation (2) except that $a_k = B_k$ and $b_{n-k} = 1$. Therefore if we calculate $B(x)e^x$ we see

$$B(x)e^x = B_0 + \left(\binom{1}{0} B_0 + \binom{1}{1} B_1 \right) \frac{x^1}{1!} + \left(\binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 \right) \frac{x^2}{2!} + \cdots$$

We can make the expression for $B(x)e^x$ look exactly like the expression that comes after the $1+$ in the expression for $B(x)$ by integrating one time. What this means is that

$$B(x) = 1 + \int B(x)e^x dx$$

or also

$$B'(x) = B(x)e^x$$

It is not trivial to solve for $B(x)$, but it is possible and if I give you the solution, it is not hard to verify that $B(x) = e^{e^x-1}$. In fact if I use "sage" to compute the Taylor expansion of e^{e^x-1}, then I see that

```
sage: taylor(exp(exp(x)-1), x, 0, 6)
203/720*x^6 + 13/30*x^5 + 5/8*x^4 + 5/6*x^3 + x^2 + x + 1
```

If I rewrite this with the $n!$ in the dominators (no simplification of the fractions) then I see that

$$e^{e^x-1} = 1 + \frac{x}{1!} + 2\frac{x^2}{2!} + 5\frac{x^3}{3!} + 15\frac{x^4}{4!} + 52\frac{x^5}{5!} + 203\frac{x^6}{6!} + \cdots$$

and this agrees with what we calculated earlier with B_0 through B_5.

I can also use sage to help me with the algebra of verifying that $B'(x) = \frac{d}{dx}(e^{e^x-1}) = e^{e^x-1}e^x = B(x)e^x$.

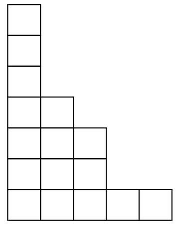
```
sage: diff(exp(exp(x)-1),x)
e^(x + e^x - 1)
sage: exp(x)*exp(exp(x)-1)
e^(x + e^x - 1)
```

I will continue to expand on the use of exponential generating functions. What we will need to do is develop tools for creating libraries of generating functions as we did for ordinary generating functions. For instance, if I give you the exponential generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$, then I expect you to be able to give me expressions for $\sum_{n\geq 0} a_{n+2}\frac{x^n}{n!}$, $\sum_{n\geq 0} n a_n \frac{x^n}{n!}$, $\sum_{n\geq 0} a_n \frac{x^{n+2}}{(n+2)!}$.

I would also like to apply our generating function techniques to objects called partitions because they are very much the type of combinatorial object where equation (1) applies, just as the recursion for the number of set partitions B_n was able to use (2).

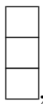
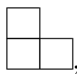
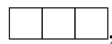
Recall that a *partition* of n is a sum $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. The order of the sum doesn't matter so to avoid confusion we assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. The λ_i are called the *parts* of the partition. r here is the number of parts of the partition or the *length* of the partition. The *sizes* of the parts are the values λ_i. The *size* of the partition is the sum of the sizes of all the parts (in this case n). Parts are called *distinct* if they are not equal to each other. The *number of parts of a given size* refers to the number of times that a value appears as a part.

A partition is represented by a diagram where I put rows of boxes and in the i^{th} row from the the bottom I put λ_i boxes and these rows of boxes are left justified. For instance the partition $(5,3,3,2,1,1,1)$ is represented as

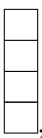
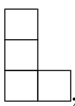
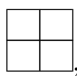
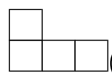
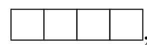


The picture is a convenient way of picturing what a partition is as a combinatorial object. Here are some examples:

Partitions of 3

, $(1,1,1)$ , $(2,1)$ , (3)

Partitions of 4

, $(1,1,1,1)$ , $(2,1,1)$ , $(2,2)$ $(3,1)$ , (4)

Lets first consider the generating function for partitions using parts of size k only. Define $\mathcal{P}_{=k}(x) = \sum_{n\geq 0}(\text{number of partitions of } n \text{ with parts of size equal to } k)x^n$. The only partitions of this type are the empty partition $()$, (k), (k,k), $(k,k,k), \ldots$. There is exactly one partition of n with parts of size k iff k divides n. Therefore the generating function is simply

$$\mathcal{P}_{=k}(x) = 1 + x^k + x^{2k} + x^{3k} + \cdots = \frac{1}{1-x^k}.$$

NOTES ON OCT 18, 2012

MIKE ZABROCKI

Last time I said that when we have a combinatorial problem like

Find the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = n$$

where $x_i \geq 0$.

We can write down the generating function to this combinatorial problem $\frac{1}{(1-x)^4}$ and when we apply restrictions of the form $x_1 + x_2$ is even and $x_3 \leq 8$. In this case the generating function for the number of solutions to this equation is (which I will not justify because we have done a number similar problems)

$$= \frac{1}{2}\left(\frac{1}{(1-x)^2} + \frac{1}{(1+x)^2}\right)\frac{1-x^9}{1-x}\frac{1}{1-x} \ .$$

Most of the combinatorial problems that we can use this method on it will be possible to reduce them to a similar enumerative question.

There is another class of problems that is useful for the Mulitplication Principle of Exponential generating functions that I discussed last time. Consider problems like:

How many words (rearrangements of the letters) in the alphabet $\{a, b, c, d\}$ are there of length n?

Since our words are of length n, there are 4^n possible words with letters in $\{a, b, c, d\}$, each letter of the word has 4 choices. The exponential generating function for the number of these words is $\sum_{n\geq 0} 4^n \frac{x^n}{n!} = e^{4x}$. But what is kind of surprising is that I can also place restrictions on the letters and write down the exponential generating function for the sequence. Say that I consider the set of words

How many words are there in the alphabet $\{a, b, c, d\}$ such that there an even number of a's and b's (total) and at most 8 c's?

If we were to enumerate this using the multiplication principle and the addition priciple, then we would choose i spots from n for the a's and b's, choose a word in the a's and b's of length i, choose j of the remaining $n - i$ for the c's such that there are at most 8

c's, then the remaining $n-i-j$ spaces are where we place the d's. By the addition and multiplication principle of generating functions, we have

(1)
$$\sum_{i+j\leq n}\binom{n}{i}\binom{\#\text{ words length } i \text{ in } a \text{ and } b}{\text{with an even } \#\ a\text{'s } \&b\text{'s}}\binom{n-i}{j}\binom{\#\text{ words of length } j}{\text{in } c \text{ with } \leq 8\ c\text{'s}}\binom{\#\text{ words of length}}{n-i-j \text{ in } d}$$

If we combine the binomials $\binom{n}{i}$ and $\binom{n-i}{j}$ and note that it is equal to $\binom{n}{i,j,n-i-j}=\binom{n}{i}\binom{n-i}{j}$.

Last time I presented the multiplication principle of exponential generating functions. I will restate it here with multiple generating functions (while the last time it was a product of two).

Principle 1. *(The Multiplication Principle of Exponential Generating Functions) Let* $A_i(x)=\sum_{n\geq 0}a_n^{(i)}\frac{x^n}{n!}$*, then*

$$A_1(x)A_2(x)\cdots A_d(x)=\sum_{n\geq 0}\left(\sum_{i_1+i_2+\cdots+i_d=n}\binom{n}{i_1,i_2,\ldots,i_d}a_{i_1}^{(1)}a_{i_2}^{(2)}\cdots a_{i_d}^{(d)}\right)\frac{x^n}{n!}\ .$$

Alternatively the coefficient of $\frac{x^n}{n!}$ *in* $A_1(x)A_2(x)\cdots A_d(x)$ *is equal to*

$$\sum_{i_1+i_2+\cdots+i_d=n}\binom{n}{i_1,i_2,\ldots,i_d}a_{i_1}^{(1)}a_{i_2}^{(2)}\cdots a_{i_d}^{(d)}\ .$$

You should recognize that (1) is a special case of a coefficient of one of these coefficients. The expression in (1) is equal to the coefficient of $x^n/n!$ in the product

(2)
$$\binom{\text{g.f. for words length } in \text{ in } a \text{ and } b}{\text{with an even } \#\ a\text{'s } \&b\text{'s}}\binom{\text{g.f. for words of length } n}{\text{in } c \text{ with } \leq 8\ c\text{'s}}\binom{\text{g.f. for words of length}}{n \text{ in } d}$$

Now I note that since there is precisely 1 word of length n using only the letter d then

$$\binom{\text{g.f. for words of length}}{n \text{ in } d}=\sum_{n\geq 0}\frac{x^n}{n!}=e^x$$

Since there is one word of length n in the letters c unless $n>8$, then

$$\binom{\text{g.f. for words of length } n}{\text{in } c \text{ with } \leq 8\ c\text{'s}}=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^8}{8!}$$

Now if we insist that there are an even number of a's and b's then the there are 4 words of length 2 (aa, ab, ba, bb), there are 16 words of length 4 ($aaaa$, $aaab$, $aaba$, …, $bbbb$). In general, the number of words of length n is 2^n if n is even and 0 if n is odd, hence the exponential generating function is equal to

$$\binom{\text{g.f. for words length } in \text{ in } a \text{ and } b}{\text{with an even } \#\ a\text{'s } \&b\text{'s}}=1+4\frac{x^2}{2!}+16\frac{x^4}{4!}+64\frac{x^6}{6!}+\cdots=\frac{1}{2}\left(e^{2x}-e^{-2x}\right)=cosh(2x)$$

Therefore putting this together with (2) we have that the coefficient of $x^n/n!$ in

$$cosh(2x)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^8}{8!}\right)e^x$$

is equal to the number of words in the alphabet $\{a, b, c, d\}$ such that there an even number of a's and b's (total) and at most 8 c's.

For example for the words of length 1 there is only c and d, for the words of length 2 we can have aa, bb, ab, ba, cc, cd, dc, dd so there are 8 words of length 2. For words of length 3 we can have caa, aca, aac, cbb, bcb, bbc, cab, acb, abc, cba, bca, bac, another 12 with a,b and ds and then 8 more are words in c and d (32 in total). In total there are We should then see that the series expands as $1 + 2\frac{x}{1!} + 8\frac{x^2}{2!} + 32\frac{x^3}{3!} + \cdots$. I will check this on the computer to show you how it is done.

```
sage: taylor(exp(x)*cosh(2*x)*sum(x^n/factorial(n) for n in range(9)),x,0,4)
16/3*x^4 + 16/3*x^3 + 4*x^2 + 2*x + 1
```

In general we have that ordinary generating functions used for counting problems that can be reduced to integer sum problems and exponential generating functions are useful for enumerating problems that can be reduced to enumerating words. It is also sometimes said that ordinary generating functions are good for enumerating "unlabeled" objects and exponential generating functions are good for enumerating "labeled" objets. This is a vague rule and hard to tell why this might be correct until we come with more examples of uses for ordinary and exponential generating functions. For example, we looked at the exponential generating function for the number of set partitions of n and this was e^{e^x-1} (this is a "labeled" object), we also started to look at partitions and ordinary generating functions.

We also talked about generating functions for partitions. I had given some of the definitions of partitions last time and I restated them. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n is a sequence of positive integers whose sum is n with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. The size of a partition is the sum of the entries $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The length of the partition is ℓ, the number of entries in the sequence.

It is difficult to give the generating function for the number partitions of n in this form because we have this condition that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$, while we know how to give the generating function for the number of non negative integer solutions to $x_1+x_2+\cdots+x_r = n$ (with potentially other conditions), but there is a way of transforming the partitions into solutions to a similar system of equations.

Let $m_i(\lambda) =$ the number of parts of λ of size i (the number of $\lambda_d = i$. Then the size of the partition λ is equal to $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell = 1m_1(\lambda) + 2m_2(\lambda) + 3m_3(\lambda) + \cdots$.

For example say that I wanted to compute the size of $(5, 2, 1, 1, 1)$. It is $10 = 5+2+1+1+1$, but since $m_1(5, 2, 1, 1, 1) = 3$, $m_2(5, 2, 1, 1, 1) = 1$, $m_3(5, 2, 1, 1, 1) = 0$, $m_4(5, 2, 1, 1, 1) = 0$, $m_5(5, 2, 1, 1, 1) = 1$ and the rest of the $m_i(5, 2, 1, 1, 1) = 0$ for $i > 5$ so the size of the partition is $1m_1(5, 2, 1, 1, 1) + 2m_2(5, 2, 1, 1, 1) + 3m_3(5, 2, 1, 1, 1) + 4m_4(5, 2, 1, 1, 1) + 5m_5(5, 2, 1, 1, 1) = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 1 = 3 + 2 + 5 = 10$.

If we look at all partitions this way we can say that all partitions are the number of solutions to the equations

$$m_1 + 2m_2 + 3m_3 + \cdots = n \tag{3}$$

with $m_i \geq 0$. Now we have phrased this question in terms of non-negative integer solutions equations and we can say that the generating function for the number of partitions of n is equal to the generating function for the number of solutions to equation (3). The generating function for the number of non-negative integer solutions to equation (3) is equal to the product of the generating functions for the number of non-negative integer solutions to $im_i = n$ over all possible $i \geq 1$. We know that the generating function for the number of non-negative solutions to the equation $im_i = n$ is equal to $\frac{1}{1-x^i}$, therefore the generating function for the number of partitions of n is equal to

$$\prod_{i \geq 1} \frac{1}{1-x^i} \ .$$

There is something a little odd about this formula because I am taking an infinite product. But because I can calculate the coefficient of x^n in this generating function by only taking the product of $\prod_{i=1}^{n} \frac{1}{1-x^i}$ (because the rest of the terms of the form $\frac{1}{1-x^{n+r}}$ for $r > 0$ don't affect the exponent of x^n), then I consider this a 'good' formula even though it seems to involve an infinite product. Since the calculation of any finite piece is finite and we can work with it (although carefully to ensure that any finite term of the series can always be computed in a finite number of steps).

Notice that if I want to compute the first 11 terms of series I just need to multiply the first 10 products together and so I can use sage to expand the series and sage also has functions which allow me to count the number of partitions of n. You should note in the code below the command `range(a,b)` are the integers i such that $a \leq i < b$ and `range(b)` are the integers $0 \leq i < b$.

```
sage: taylor(prod(1/(1-x^i) for i in range(1,11)),x,0,10)
42*x^10 + 30*x^9 + 22*x^8 + 15*x^7 + 11*x^6 + 7*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
sage: [Partitions(n).cardinality() for n in range(0,11)]
[1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42]
```

We then considered odd partitions, that is, partitions where all entries in the parts are odd. The number of odd partitions of n is equal to the number of non-negative integer solutions to the equation:

$$1m_1 + 3m_3 + 5m_5 + \cdots = n.$$

Leaving out the argument this time (because it seems we have done it so many times), the generating function for the number of odd partitions of n is equal to

$$\prod_{i \geq 0} \frac{1}{1-x^{2i+1}}$$

Again I can use sage to compute both the taylor series for the first 10 or so terms and use it to count the number of odd partitions of n. In the following snippit of code, I compute the Taylor series for the generating function and I also compute the partitions of n and then I restrict (filter) them so that I look at the ones where all entries are odd.

```
sage: taylor(prod(1/(1-x^(2*i+1)) for i in range(0,5)),x,0,10)
10*x^10 + 8*x^9 + 6*x^8 + 5*x^7 + 4*x^6 + 3*x^5 + 2*x^4 + 2*x^3 + x^2 + x + 1
sage: [Partitions(n).filter(lambda x: all(mod(v,2)==1 for v in x)).cardinality()
...: for n in range(0,11)]
[1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10]
```

Next we looked at strict partitions or partitions with distinct parts. A partition is called *strict* if there is at most one part of any given size (or otherwise stated, no parts are repeated). If we phrase this in terms of solutions to equations we would consider equations of the form

$$m_1 + 2m_2 + 3m_3 + \cdots = n$$

with $0 \leq m_i \leq 1$. The restriction that the parts are distinct (or the partition is strict) imposes the condition that m_i is either 0 or 1 since m_i represents the number of parts of size i. Again, without further explation the generating function for the number of solutions to these equations is

$$\prod_{i\geq 1}(1 + x^i)$$

Again I can use sage to calculate both the series and the number of such partitions. This time I looked in the documentation in order to find the number of partitions of n with distinct parts and it said the command is: `Partitions(n, max_slope=-1).cardinality()`.

```
sage: taylor(prod(1+x^i for i in range(1,11)),x,0,10)
10*x^10 + 8*x^9 + 6*x^8 + 5*x^7 + 4*x^6 + 3*x^5 + 2*x^4 + 2*x^3 + x^2 + x + 1
sage: [Partitions(n, max_slope=-1).cardinality() for n in range(11)]
[1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10]
```

Hmmm, I wonder if there is a connection between the number of strict partitions and the number of odd partitions?

Then I gave you a worksheet (which I will attach).

NOTES ON OCT 23, 2012

MIKE ZABROCKI

$$1 + a + a^2 + a^3 + a^4 + \cdots = \frac{1}{1-a}$$

$$1 + a + a^2 + a^3 + \cdots + a^r = \frac{1-a^{r+1}}{1-a}$$

Last time we finished by looking at the matching worksheet of generating functions of sets of partitions. I want to move beyond "recognizing" when one generating function expression is a generating function for the number of partitions of a certain type to "deriving" the generating function expression for a set of partitions. Partitions because of the way that partitions are made up, they are sets of objects that are well suited for expressing the generating functions for the number of objects with algebraic expressions. This is not possible with most sets of combinatorial objects.

The study of partitions as combinatorial objects is often considered as part of the domain number theory since a partition n is a way of writing n as a sum of integers.

I gave the answers for the worksheet that I posted. I got very few questions about the answers but someone asked how to explain the generating function for the partitions of n with even parts and at most 4 parts of any given size. So I started to break down this set of partitions in two different ways.

Method 1: notice that if I let m_i be the number of parts of size i then every partition with even parts and at most 4 parts of any given size is a solution to the integer equation

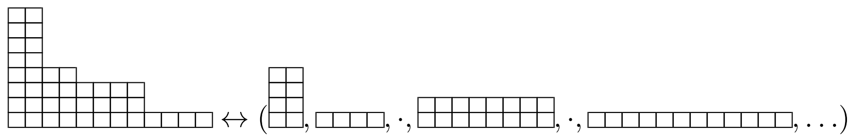
$$2m_2 + 4m_4 + 6m_6 + \cdots = n$$

where $0 \leq m_i \leq 4$. The generating function for this set of solutions is the product of the generating functions for the number of solutions to $2rm_{2r} = n$ with $0 \leq m_{2r} \leq 4$ for $r \geq 1$. We know that the generating function for the number of solutions to $2rm_{2r} = n$ with $0 \leq m_{2r} \leq 4$ is $1 + x^{2r} + x^{4r} + x^{6r} + x^{8r} = \frac{1-x^{10r}}{1-x^{2r}}$. Hence the generating function for the number of partitions with even parts and at most 4 parts of any given size is equal to $\prod_{r\geq 1} \frac{1-x^{10r}}{1-x^{2r}}$.

Method 2: I can break down this set of partitions into component pieces as a picture. Imagine that a partition partitions with even parts and at most 4 parts of any given size consists of at most 4 parts of size 2, at most 4 parts of size 4, at most 4 parts of size 6, etc.

In fact, a partition can be decomposed into a tuple consisting of parts of size $2r$ for $r \geq 1$ and there can be 0,1,2,3,or 4 parts of size $2r$.

For example the partition $(12, 8, 8, 4, 2, 2, 2, 2)$ can be decomposed into a tuple consisting of the parts $((2, 2, 2, 2), (4), (), (8, 8), (), (12), \ldots)$, or graphically



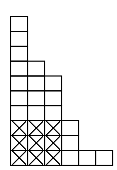
Again, I know that the generating function for the parts of size $2r$ is $1 + x^{2r} + x^{4r} + x^{6r} + x^{8r} = \frac{1-x^{10r}}{1-x^{2r}}$. Whenever we have a set of tuples like this we can apply the multiplication principle of generating functions and hence the generating function for the number of partitions of n with even parts and at most 4 parts of any given size is equal to $\prod_{r \geq 1} \frac{1-x^{10r}}{1-x^{2r}}$.

Using the same reasoning as in the above example and that we used in the last class, the generating function for the number of partitions of n with parts of size equal to k is $\mathcal{P}_{=k}(x) = \frac{1}{1-x^k}$. The generating function for the number of partitions of n with parts of size $\leq k$ is $\mathcal{P}_{\leq k}(x) = \prod_{i=1}^{k} \frac{1}{1-x^i}$.

We also said that the generating function for the partitions of n with no restriction is $\prod_{i \geq 1} \frac{1}{1-q^i}$ (this works by taking the limit of $\mathcal{P}_{\leq k}(x)$ as $k \to \infty$ and ensuring that for every coefficient of x^n that we might want to calculate is the same for $k > n$).

I also want to consider the partitions of length precisely equal to k, or alternatively if I take the transpose of these diagrams, this is the set of partitions whose first part is exactly equal to k. Every partition whose first part is exactly equal to k is isomorphic to a pair (X, Y) where X is some number ≥ 1 of parts of size equal to k and Y is a partition whose parts of size $\leq k - 1$. The generating function for the partitions consisting of at least one part of size of size k is equal to $x^k + x^{2k} + x^{3k} + \cdots = \frac{x^k}{1-x^k}$. Therefore the generating function for the number of partitions whose first part is exactly equal to k is equal to $\frac{x^k}{1-x^k}\mathcal{P}_{\leq k-1}(x) = x^k \prod_{i=1}^{k} \frac{1}{1-x^i}$.

A Durfee square is the largest square that can fit in the diagram for a partition. For example, if my partition is $(6, 4, 4, 3, 3, 3, 2, 1, 1, 1)$ then the diagram of the partition is



and I can't put a larger square than 3×3 in that diagram. Now you can see from the example that I have drawn here that sitting on top of that Durfee square is a partition whose largest part is at most 3, and sitting off to the right of the Durfee square is a partition whose length is at most 3.

In general we can say that every partition that contains a $k \times k$ Durfee square is isomorphic to (a $k \times k$ Durfee square, a partition whose largest part is at most k, a partition whose length is at most k). By transposing a partition whose length is at most k, we have a partition whose largest part is at most k, therefore

g.f. for the number of partitions of n whose largest part is at most $k =$

g.f. for the number of partitions of n whose length is at most $k = \mathcal{P}_{\leq k}(x)$

By the MPofGFs, the generating function for partitions with a Durfee square equal to k is equal to

$$x^{k^2}\mathcal{P}_{\leq k}(x) = x^{k^2}\prod_{i=1}^{k}\frac{1}{(1-x^i)^2} \ .$$

Now if I also remark that every partition is either empty, or contains a Durfee square of size k for some $k \geq 1$, then I see that the generating function for all partitions (by the addition principle of generating functions) is equal to

$$= 1 + \sum_{k\geq 1} x^{k^2}\prod_{i=1}^{k}\frac{1}{(1-x^i)^2}$$

But we already knew that this was equal to an infinite product so we have shown the algebraic relation

$$\prod_{i\geq 1}\frac{1}{1-x^i} = 1 + \sum_{k\geq 1} x^{k^2}\prod_{i=1}^{k}\frac{1}{(1-x^i)^2} \ .$$

In case this is hard to comprehend, I will compute it on the computer and show you that the series are the same (at least for the first few terms.

```
sage: prod(1/(1-x^i) for i in range(1,10))
-1/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 - 1)*(x^6 - 1)*(x^7 - 1)*(x^8 - 1)*(x^9 - 1))
sage: taylor(prod(1/(1-x^i) for i in range(1,10)),x,0,10)
41*x^10 + 30*x^9 + 22*x^8 + 15*x^7 + 11*x^6 + 7*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
sage: taylor(1+x/(1-x)^2+x^4/((1-x)*(1-x^2))^2+x^9/((1-x)*(1-x^2)*(1-x^3))^2,x,0,10)
42*x^10 + 30*x^9 + 22*x^8 + 15*x^7 + 11*x^6 + 7*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
```

Notice that these two series differ in exactly the coefficient of x^{10}. This is because my first series is only the product of the terms $\frac{1}{1-x^i}$ for $1 \leq i < 10$ and so these two series will differ after the 10^{th} term.

I can also remark that every partition is empty or it has length equal to k for some $k \geq 1$. This implies that the generating function for the number of partitions of n is equal to (by my argument on p.2 of these notes,

$$1+\sum_{k\geq 1} x^k \prod_{i=1}^{k} \frac{1}{1-x^i}$$

This is a very powerful tool now that we have developed it properly, because we have shown that

$$\prod_{i\geq 1} \frac{1}{1-x^i} = 1+\sum_{k\geq 1} x^{k^2} \prod_{i=1}^{k} \frac{1}{(1-x^i)^2} = 1+\sum_{k\geq 1} x^k \prod_{i=1}^{k} \frac{1}{1-x^i}$$

in other words, that an infinite product is equal to two different infinite sums just by arguing with pictures. Lets verify that this last sum is the same by calculating the example with the computer.

```
sage: f = 1+sum(x^i/prod(1-x^j for j in range(1,i+1)) for i in range(1,10))
sage: f
-x^9/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 - 1)*(x^6 - 1)*(x^7 -
1)*(x^8 - 1)*(x^9 - 1)) + x^8/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 -
1)*(x^6 - 1)*(x^7 - 1)*(x^8 - 1)) - x^7/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 -
1)*(x^5 - 1)*(x^6 - 1)*(x^7 - 1)) + x^6/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 -
1)*(x^5 - 1)*(x^6 - 1)) - x^5/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)*(x^5 -
1)) + x^4/((x - 1)*(x^2 - 1)*(x^3 - 1)*(x^4 - 1)) - x^3/((x - 1)*(x^2 -
1)*(x^3 - 1)) + x^2/((x - 1)*(x^2 - 1)) - x/(x - 1) + 1
sage: taylor(f,x,0,10)
41*x^10 + 30*x^9 + 22*x^8 + 15*x^7 + 11*x^6 + 7*x^5 + 5*x^4 + 3*x^3 + 2*x^2 + x + 1
```

Again, this series is wrong in the coefficient of x^{10} because I didn't add enough terms from my series, but I can easily change how many terms I add together and compute this series as high as I need.

I then gave you a worksheet where you were asked to do something similar to what I just did by giving an expression for the generating function for certain sets of partitions. I gave you each a problem from this and I really wanted everyone to go home and think about *one* problem. I then said that I would pick one at random that I would solve. I think that the next one that was available was (9) and when I looked at it I realized that the answer was complicated (I didn't know how to solve it). I made up these problems and some times it is possible to write down a sentence where the answer to that question is not 'nice.' This was one of those. I just had to change a few words and it corrected the problem to something that is solvable. The version that is on the website had the corrected version. Instead I solved number (10) in class and asked you to think about your question.

The instructions read: Apply the addition or the multiplication principle of generating functions to give the generating function for the following sequences of numbers.

(10) the number of partitions of n with with odd parts and a part will either occur 0 or an odd number of times

We decompose the partitions of n with odd parts that will occur 0 or an odd number of times into a tuple consisting of the parts of size 1, 3, 5, etc. Hence we can apply the MPofGFs to take the product for $i \geq 0$ of the parts of size $2i+1$ which occur 0 or an odd number of times. The generating function for those parts of size $2i+1$ which occur 0 or an odd number of times is equal to

$$\begin{aligned}
&1 + x^{2i+1} + x^{3(2i+1)} + x^{5(2i+1)} + x^{7(2i+1)} + \cdots = \\
&1 + x^{2i+1}(1 + x^{2(2i+1)} + x^{4(2i+1)} + x^{6(2i+1)} + \cdots) = \\
&1 + \frac{x^{2i+1}}{1 - x^{2(2i+1)}}
\end{aligned}$$

Therefore the generating function for the number of partitions of n with odd parts that will occur 0 or an odd number of times is equal to

$$\prod_{i \geq 0} \left(1 + \frac{x^{2i+1}}{1 - x^{2(2i+1)}}\right)$$

NOTES ON OCT 25, 2012

MIKE ZABROCKI

I wanted you do the problems on the worksheet that I gave you last time. Only a few people had done their problem. Even if it was a matter of just trying, it on the board so that we can see what was right and what was wrong, this was a good thing. We had a few people put up their answers:

(5) the number of partitions of n with at most 8 parts of any given size.
(28) the number of partitions of n with Durfee square of size 3×3 and all even parts.
(32) the number of partitions of n with a Durfee square of even size and all parts even.
hmmm...there was one more but it is 4 days later and I can't remember which one it was.

Someone asked me if I could post the answers and I agreed reluctantly that I would post the answers to some them. I am rescinding that statement. I will post the solutions/answers to any that people agree to present a solution to in class. I will check any answers that people want to verify with me through email. But if I post the answers, then this question becomes an entirely different problem. Rather than learning how to derive the answers yourself, you only have to match your answer/explanation against my expression. The matching worksheet already has a bunch of 'descriptions' and 'expressions' so if you need examples, then you have 18 of them right there. Here are three more right here.

The instructions read: Apply the addition or the multiplication principle of generating functions to give the generating function for the following sequences of numbers.

(5) the number of partitions of n with at most 8 parts of any given size.

The generating function for the partitions consisting only of parts of size i with at most 8 parts is equal to

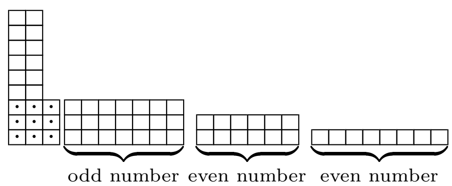
$$1 + x^i + x^{2i} + \cdots + x^{8i} = \frac{1 - x^{9i}}{1 - x^i}$$

The generating function for the number of partitions of n with at most 8 parts of any given size will be the product of the generating functions of the partitions consisting only of parts of size i with at most 8 parts for $i \geq 1$ because each partition can be decomposed into the parts of size i for $i \geq 1$. Therefore the generating function is equal to

$$\prod_{i \geq 0} \frac{1 - x^{9i}}{1 - x^i}$$

(28) the number of partitions of n with Durfee square of size 3×3 and all even parts.

A partition with a 3×3 Durfee square and all parts even consists of (a 3×3 Durfee square, a partition which lies above the Durfee square consisting only of parts of size 2, a partition that lies to the right of the Durfee square consisting of exactly three parts and all parts odd). The third entry in this tuple can be also be described as a partition consisting of an odd number of columns of size 3, an even number of columns of size 2 and an even number of columns of size 1.



This decomposition of a partition into these pieces implies that we can apply the MPofGFs and the the generating function for this whole set of partitions is equal to the product of the generating function for partitions with parts of size 2 only $= \frac{1}{1-x^2}$, the generating function for a 3×3 Durfee square $= x^3$, the generating function for an odd number of columns of size 3 $x^3 + x^9 + x^{15} + x^{21} + \cdots = \frac{x^3}{1-x^6}$, the generating function for the even number of columns of length $2 = 1 + x^4 + x^8 + x^{12} + \cdots = \frac{1}{1-x^4}$, the generating function for an even number of columns of length $1 = \frac{1}{1-x^2}$. Therefore the generating function for the number of partitions of n with Durfee square of size 3×3 and all even parts is equal to

$$\frac{1}{1-x^2} x^3 \frac{x^3}{(1-x^2)(1-x^4)(1-x^6)} = \frac{x^6}{(1-x^2)^2(1-x^4)(1-x^6)}$$

(32) the number of partitions of n with a Durfee square of even size and all parts even

A partition of n with a Durfee square of size $2k$ and all parts even consists of (a Durfee square of size $2k \times 2k$, a partition which lies above the Durfee square with all parts even and maximum part $2k$, a partition which lies to the right of the Durfee square where all parts are even and the length is less than or equal to $2k$). A "partition where all parts are even and the length is less than or equal to $2k$" can also be described as some even number of columns of size i for $1 \leq i \leq 2k$. Since the generating function for a even number of columns of size i is $\frac{1}{1-x^{2i}}$ hence the generating function for the partitions which lie to the right of the $2k \times 2k$ Durfee square is equal to $\prod_{i=1}^{2k} \frac{1}{1-x^{2i}}$. The partitions which are above the Durfee square consist only of even parts between 1 and $2k$, hence by the MPofGFs the generating function for the partitions which lies above the Durfee square with all parts even and maximum part $2k$ is equal to $\prod_{i=1}^{k} \frac{1}{1-x^{2i}}$. The Durfee square itself has generating function x^{4k^2}. Hence the generating function for partitions of n with a Durfee square of size $2k$ and all parts even is

$$x^{4k^2} \prod_{i=1}^{k} \frac{1}{1-x^{2i}} \prod_{i=1}^{2k} \frac{1}{1-x^{2i}} .$$

Now since all partitions of n with a Durfee square of even size and all parts even are either the empty partition or have a Durfee square of size $2k \times 2k$ for $k \geq 1$, then the generating function is

$$1 + \sum_{k \geq 1} x^{4k^2} \prod_{i=1}^{k} \frac{1}{1-x^{2i}} \prod_{i=1}^{2k} \frac{1}{1-x^{2i}} .$$

I expect you do the rest of these problems on your own. You won't learn any more by just reading. You have to learn to figure these out yourself.

The next thing that we are going to cover is Pólya enumeration. This requires that we know what the concept of a group is. If you have had a course in algebra before you have likely encountered the definition of a group before. You have all encountered the concept of a group. Let me tell you what one is and then show you that you have lots of examples:

A group is a set of elements G (possibly finite, possibly infinite) with a binary operation denoted $*$. That is $* : G \times G \to G$ and usually we denote it as $a * b \in G$ for $a, b \in G$. There are a few properties that this binary operation has in order to be a group.

(1) The product is associative, that is, for $a, b, c \in G$, $a * (b * c) = (a * b) * c$.
(2) There is an element $e \in G$ such that $g = e * g = g * e$ for all $g \in G$.
(3) For each element in $a \in G$, there is another element $\overline{a} \in G$ (called the inverse of a) such that $a * \overline{a} = \overline{a} * a = e$ (in many cases, we write the element $\overline{a} = a^{-1}$ but just remember that this does not mean $1/a$).

Here are some examples that you are probably familiar with:

(1) The integers $\mathbb{Z}$ with the binary operation of $+$. This example has the identity element 0 because $0 + a = a + 0 = a$ for all $a \in \mathbb{Z}$. For every integer a, $\overline{a} = -a$ has the property that $a + \overline{a} = 0$. Also addition is associative.
(2) The rational numbers except 0, $\mathbb{Q}\backslash\{0\}$, with multiplication $\cdot$ is the binary operation is an example of a group. In this example the identity element is 1 because $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{Q}\backslash\{0\}$. Moreover if $a \in \mathbb{Q}\backslash\{0\}$, then $\overline{a} = 1/a$ is an element such that $a\overline{a} = \overline{a}a = 1$. Also multiplication is associative.

(3) The group of permutations of 3, $G = \{123, 132, 213, 231, 312, 321\}$, with $a_1a_2a_3 \circ b_1b_2b_3 = b_{a_1}b_{a_2}b_{a_3}$. This example is a little different than the other examples because it is not immediately familiar to us that the multiplication is associative. In fact, it is since

$$(a_1a_2a_3 \circ b_1b_2b_3) \circ c_1c_2c_3 = b_{a_1}b_{a_2}b_{a_3} \circ c_1c_2c_3 = c_{b_{a_1}}c_{b_{a_2}}c_{b_{a_3}}$$
$$a_1a_2a_3 \circ (b_1b_2b_3 \circ c_1c_2c_3) = a_1a_2a_3 \circ c_{b_1}c_{b_2}c_{b_3}$$

and if you understand this properly, you can see that these are the same thing. Now the identity of this group is the element 123 since $123 \circ b_1b_2b_3 = b_1b_2b_3$. It is also the case that $a_1a_2a_3 \circ 123 = a_1a_2a_3$. You can check that the inverse element exists for for each of the 6 permutations in this group. Check that 123, 132, 213 and 321 are equal to their own inverse, 231 and 312 are inverses of each other.

OK these are three examples of groups and kind of cover a small range of examples, but groups are everywhere. In order to understand a definition clearly it is also a good idea to try to understand an example of something which is not a group.

(1) Take for example the integers except 0, $\mathbb{Z}\backslash\{0\}$, with the binary operation of $\cdot$ multiplication. This is an example of something which is not a group because there is nothing you can multiply the element 2 by in order to get 1 so there is no inverse of the element 2 (well, you can multiply it by $1/2$, but that isn't an integer and this is why $\mathbb{Q}\backslash\{0\}$ is a group and $\mathbb{Z}\backslash\{0\}$ is not).
(2) None of the integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ are groups with multiplication as the operation since they all include 0 and there is nothing you can multiply 0 by and get 1 (the identity element of the group). You might ask, what happens if I "throw in infinity and then define $0 \cdot \infty = \infty \cdot 0 = 1$" This is a great idea but it just kicks the problem to somewhere else in your group since $\infty \cdot (0 \cdot 2) = \infty \cdot 0 = 1$, but $(\infty \cdot 0) \cdot 2 = 1 \cdot 2 = 2$. It is the case that all of $\mathbb{Q}\backslash\{0\}$, $\mathbb{R}\backslash\{0\}$, $\mathbb{C}\backslash\{0\}$ are groups with multiplication as the binary operation, but if they include 0 then they are not a group.

NOTES ON OCT 30, 2012

MIKE ZABROCKI

I started with asking who was willing to put up their solution for their problem of a generating function for a set of partitions of n. Rachel volunteered and this is the solution that we eventually came up with.

(4) the number of partitions of n with parts of size 1, 2 or 3 occurring at most 8 times each.

Every partition of n with parts of size 1, 2 or 3 occurring at most 8 times each can be decomposed into ≤ 8 parts of size 1, ≤ 8 parts of size 2, ≤ 8 parts of size 3, therefore the generating function for partitions of n with parts of size 1, 2 or 3 occurring at most 8 times each is equal to

$$\prod_{i=1}^{3} (\text{generating function for partitions of } n \text{ with parts of size } i \text{ occurring at most 8 times})\,.$$

The generating function for partitions of n with parts of size i occurring at most 8 times is equal to

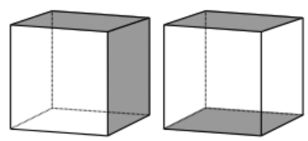
$$1 + x^i + x^{2i} + \cdots + x^{8i} = \frac{1 - x^{9i}}{1 - x^i}$$

and therefore the generating function for partitions of n with parts of size 1, 2 or 3 occurring at most 8 times each is equal to

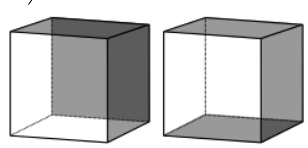
$$\frac{(1 - x^9)(1 - x^{18})(1 - x^{27})}{(1 - x)(1 - x^2)(1 - x^3)}\;.$$

I tried to add some details about the problem you were asked to do for homework. Note that the number of odd partitions of 4 is equal to 2 because only $(3, 1)$ and $(1, 1, 1, 1)$ are the only two partitions with odd parts of size 4. The problem that you are asked to compute for the problem in the homework is the exponential generating function for the odd *set* partitions. The odd partitions of 4 are different than the odd set partitions of $\{1, 2, 3, 4\}$. There are 5 of odd set partitions (where all parts of odd size) of $\{1, 2, 3, 4\}$ that are given by $\{\{1, 2, 3\}, \{4\}\}$, $\{\{1, 2, 4\}, \{3\}\}$, $\{\{1, 3, 4\}, \{2\}\}$, $\{\{2, 3, 4\}, \{1\}\}$, $\{\{1\}, \{2\}, \{3\}, \{4\}\}$. For the recurrence on the coefficients, they satisfy $B^{odd}_0 = 1$, $B^{odd}_1 = 1$, $B^{odd}_2 = \binom{1}{0} B^{odd}_1 = 1$, $B^{odd}_3 = \binom{2}{0} B^{odd}_2 + \binom{2}{2} B^{odd}_0 = 1 + 1 = 2$, $B^{odd}_4 = \binom{3}{0} B^{odd}_3 + \binom{3}{2} B^{odd}_1 = 2 + 3 \cdot 1 = 5$. I am not asking you to show that B^{odd}_n is equal to the number of odd set partitions of n (you should be able to do this, but that is a different problem) but we see that this agrees for $n = 4$ and for $n = 1$ the only set partition is $\{\{1\}\}$, for $n = 2$ the only odd set partition is $\{\{1\}, \{2\}\}$, for $n = 3$ there are two set partitions $\{\{1, 2, 3\}\}$ and $\{\{1\}, \{2\}, \{3\}\}$.

I wanted to motivate what we are going to do with groups a bit so I posed the following problem. How many ways are there of coloring the faces of the cube with 2 black faces and 4 white faces? Immediately someone answered $\binom{6}{2}$ and while this is correct, I wrote this down as 'Answer 1,' because there is a way of thinking of this problems such that there is a different answer. Certainly if the faces of the cube are all numbered and all distinct then there are $\binom{6}{2}$ ways of coloring the faces, but if all the faces are identical and we are allowed to rotate the cube then there are 2 ways of coloring the cube, either the two black faces are next to each other or they are on opposite sides of the cube. This is my 'Answer 2.'
Answer 1: $\binom{6}{2}$
Answer 2: 2



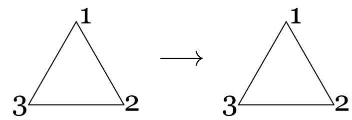
I then asked the same question if we color the cube with three black faces and three white faces.
Answer 1: $\binom{6}{3}$
Answer 2: 2 (either all three black faces share two edges or only one of the faces is shares two black edges : see the diagram)

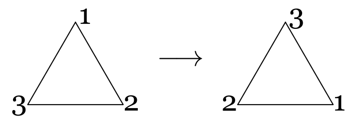


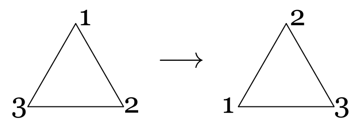
Hopefully these diagrams are clear enough to tell the difference between the two. I then suggested that we write down the generating function for the number of colorings with k black faces and $6 - k$ while faces.
Answer 1: $\binom{6}{0}B^0W^6 + \binom{6}{1}B^1W^5 + \binom{6}{2}B^2W^4 + \binom{6}{3}B^3W^3 + \binom{6}{4}B^4W^2 + \binom{6}{5}B^5W^1 + \binom{6}{6}B^6W^0$
Answer 2: $B^0W^6 + B^1W^5 + 2B^2W^4 + 2B^3W^3 + 2B^4W^2 + B^5W^1 + B^6W^0$

For answer 1 we should recognize that this is exactly $(W + B)^6$, but it isn't obvious what the second generating function formula is. What we will do in the next few weeks is develop the techniques which will give a formula for both of these generating functions such that they are both special cases. The reason I introduced the notion of a group last time is that we will use groups in our formula. The reason is that the set of motions of a shape form a group so I set up some examples and notation for looking motions of a shape.

I want to explain what the motions of a cube are. For this we need to set up some notation. Lets start with a much smaller example like the motions of a triangle and I want to indicate how it is an example of a group. Consider a triangle with labeled vertices and look at just the rotations of the triangle:

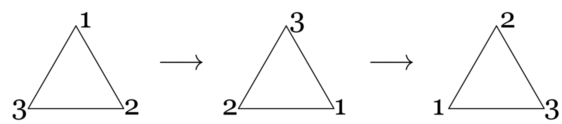
$identity = e$: 

R_{120} : 

R_{240} : 

The names that I have given to these operations are slightly misleading, because in a minute I am going to define them more precisely. The identity has the effect of doing nothing. The operation R_{120} takes the vertex 1 and sends it to 3, takes the vertex 3 and changes it to 2, takes the vertex labeled with 2 and changes it to 1. The operation R_{240} is the operation which takes the vertex 1 and changes it to a 2, takes the vertex 2 and changes it to a 3, takes the vertex 3 and changes it to a 1.

I noticed that if you do two operations of R_{120} then you obtain the same result as if you do one R_{240}.

R_{120} : 

Simlarly, if you do two R_{240} operations then you get the same effect as a R_{120}. So what we do is define a binary operation which is composition of these operations and set $R_{120} \circ R_{120} = R_{240}$ and $R_{240} \circ R_{240} = R_{120}$ and $R_{120} \circ R_{240} = R_{240} \circ R_{120} = e$. I can make a 'multiplication table' for these operations as follows

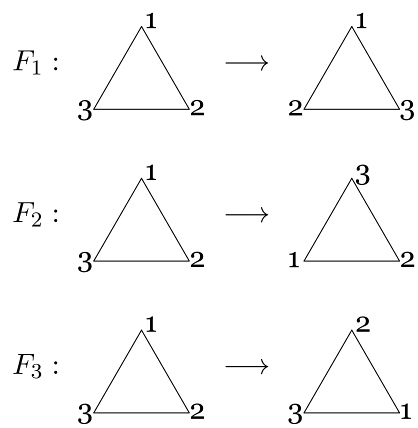
$\circ$	e	R_{120}	R_{240}
e	e	R_{120}	R_{240}
R_{120}	R_{120}	R_{240}	e
R_{240}	R_{240}	e	R_{120}

This is an example of a group. You can verify by checking on all the elements of the set $\{e, R_{120}, R_{240}\}$ that all of the conditions needed for this to be a group are satisfied with the operation of $\circ$ (see the definition from the notes on October 25). One thing that I plan to show at a later date is that in a multiplication table for a group, each element of the group appears exactly once in each row and each column.

But this is not the only group that we can make with the motions of a triangle because I can also flip the

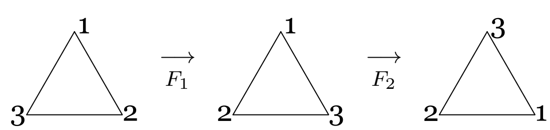
At this point I introduced notation which allowed me to use a shorthand for these operations and it is called *cycle notation*. When I write $R_{120} = (132)$, then I mean that "the vertex 1 is sent to 3, the vertex 3 is sent to 2, the vertex labeled by 2 is sent to 1 (the first entry in my cycle)." Using this same notation $R_{240} = (123)$ because as we said before "under the operation R_{240}, the vertex 1 is changed to the vertex 2, the vertex 2 is changed to the vertex 3 and the vertex 3 is changed to the vertex 1." Then, to give notation to the identity element I will say that $e = (1)(2)(3)$ because "the vertex 1 is 'changed' to the vertex 1, the vertex 2 is changed to the vertex 2, and the vertex 3 is changed to the vertex 3."

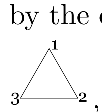
Then I introduced three more operations:



I want to represent them in my cycle notation as $F_1 = (1)(23)$ because "vertex 1 is fixed, vertex 2 is sent to vertex 3 and vertex 3 is sent to vertex 2." $F_2 = (2)(13)$ because "vertex 2 is fixed, vertex 1 is sent to vertex 3 and vertex 3 is sent to vertex 1." $F_3 = (12)(3)$ because "vertex 3 is fixed, vertex 1 is sent to vertex 2 and vertex 2 is sent to vertex 1."

If we do any of the F_1 operations twice then we get back to the original shape so $F_1 \circ F_1 = F_2 \circ F_2 = F_3 \circ F_3 = e$. If we do an F_1 operation followed by an F_2 then we have



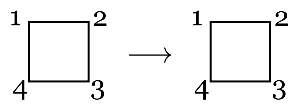
You should pay close attention to the second arrow and what is meant by that. This should resolve a question of what I mean by the operation of $\longrightarrow$. When I wrote the definition of F_2 and how it acts on the picture , it doesn't completely resolve what I mean when I act the operation of F_2 on another picture. I have to choose a convention because what I want it to mean is that F_2 leaves the vertex labelled by 2 alone and the vertex labeled by 1 is changed so that it is labelled by 3 and the vertex labelled by 3 is changed so that it is afterwards labelled by 1. The action of $F_2 \circ F_1($ triangle $) = F_2(F_1($ triangle $)) = R_{120}($ triangle $)$. Therefore we say that $F_2 \circ F_1 = R_{120}$.

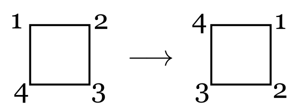
It turns out that the set $\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$ also forms a group. You will have to check it has the following multiplication table by doing the individual compositions of the operations.

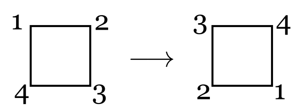
$\circ$	e	R_{120}	R_{240}	F_1	F_2	F_3
e	e	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_{240}	e	F_2	F_3	F_1
R_{240}	R_{240}	e	R_{120}	F_3	F_1	F_2
F_1	F_1	F_3	F_2	e	R_{240}	R_{120}
F_2	F_2	F_1	F_3	R_{120}	e	R_{240}
F_3	F_3	F_2	F_1	R_{240}	R_{120}	e

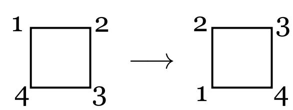
I didn't mention that $\{e, F_1\}$, $\{e, F_2\}$ and $\{e, F_3\}$ are also all groups of motions of the triangle. They all satisfy (0) if $x, y \in G$, then $x \circ y \in G$, (1) there is an e in G such that $e \circ x = x \circ e = x$ for all $x \in G$, (2) for each $x \in G$ there is an $x^{-1} \in G$ such that $x \circ x^{-1} = x^{-1} \circ x = e$ and (3) for all $x, y, z \in G$, $x \circ (y \circ z) = (x \circ y) \circ z$.

I also said that we should next look at the operations that we can do on a square because this example is at least a little larger and we might be able to see some subtleties that we cannot see on the triangle. There are 4 'rotations' which I drew as:

$e = (1)(2)(3)(4)$:


$R_{90} = (1432)$:


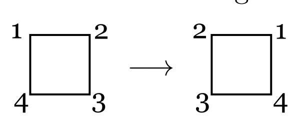
$R_{180} = (13)(24)$:


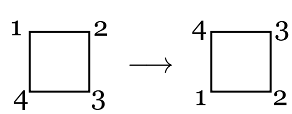
$R_{270} = (1234)$:


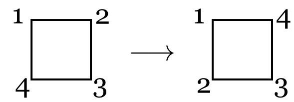
The 'multiplication table' for this group looks like

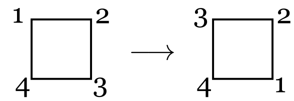
$\circ$	e	R_{90}	R_{180}	R_{270}
e	e	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	e
R_{180}	R_{180}	R_{270}	e	R_{90}
R_{270}	R_{270}	e	R_{90}	R_{180}

If we allow flipping this square then there are 4 more operations that involve flipping across the vertical, the horizontal and across either of the two diagonals.

$V = (12)(34)$:


$H = (14)(23)$:


$D_1 = (1)(24)(3)$: 

$D_2 = (13)(2)(4)$: 

I recommend that for practice that you build the 8×8 multiplication table for this group. It is good idea to try it to make sure that you understand the the operations and the notation that we have introduced here.

NOTES ON NOV 6, 2012

MIKE ZABROCKI

Last time we had some examples of groups:
Motions of a triangle with rotations only $\{e, R_{120}, R_{240}\}$
Motions of a triangle with rotations and flips $\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$
Motions of a square with rotations only $\{e, R_{90}, R_{180}, R_{270}\}$
Motions of a square with rotations and flips $\{e, R_{90}, R_{180}, R_{270}, F_H, F_V, F_{D_1} F_{D_2}\}$

Another good example of a group is the set $\{0, 1, 2, \ldots, n-1\}$ and the operation of addition *mod* n. If $n = 3$, the set of elements is $\{0, 1, 2\}$ and the operation is addition *mod* 3. The multiplication table looks like

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

I claim that we have seen this group before. Take a look at the following table from the last lecture:

$\circ$	e	R_{120}	R_{240}
e	e	R_{120}	R_{240}
R_{120}	R_{120}	R_{240}	e
R_{240}	R_{240}	e	R_{120}

It is the 'same' in some sense. What does it mean when I say that the groups are the same? I mean that there is a relabeling of the multiplication tables so that they are the same.

We say that a map f from a group $(G_1, *)$ to a group $(G_2, \cdot)$ is called a homomorphism if

$$f(g * h) = f(g) \cdot f(h) \tag{1}$$

for all g and h in G_1. If f is a bijection, then G_1 and G_2 are said to be isomorphic groups.

In the example above, we take $f(0) = e$, $f(1) = R_{120}$ and $f(2) = R_{240}$. Under this map, the tables look exactly the same and this is what is meant by equation (1).

Example 2:
the table of addition *mod* 4 looks like the following.

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

I claim that we also saw this table the other day when we had the group table

$\circ$	e	R_{90}	R_{180}	R_{270}
e	e	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	e
R_{180}	R_{180}	R_{270}	e	R_{90}
R_{270}	R_{270}	e	R_{90}	R_{180}

These tables are the 'same' because we can find a map $f(0) = e$ (the identity of each group always goes to the identity in a homomorphism), $f(1) = R_{90}$, $f(2) = R_{180}$, $f(3) = R_{270}$.

Example 3:
Consider the group $\{e, F_V\}$ with the multiplication table that looks like

$\circ$	e	F_V
e	e	F_V
F_V	F_V	e

Consider the map from $(\{e, F_V\}, \circ)$ to $(\{e, R_{90}, R_{180}, R_{270}\}, \circ)$ by the map $f(e) = e$ and $f(F_V) = R_{180}$. This map is a group homomorphism, because the group consisting of $(\{e, R_{180}\}, \circ)$ has the same multiplication table as $(\{e, F_V\}, \circ)$.

Example 4:
Consider the map from $(\{e, R_{90}, R_{180}, R_{270}\}, \circ)$ to $(\{e, F_V\}, \circ)$ such that $f(e) = f(R_{180}) = e$ and $f(R_{90}) = f(R_{270}) = F_V$. If we look at the image of the multiplication table in Example 2 and apply the map to it, we see

$\circ$	e	F_V	e	F_V
e	e	F_V	e	F_V
F_V	F_V	e	F_V	e
e	e	F_V	e	F_V
F_V	F_V	e	F_V	e

And this is all in agreement with the table from Example 3 (above).

Example 5:
We can also define a map $(\{0, 1, 2, 3\}, + \ mod\ 4)$ into itself that sends $f(i) = 2i \ mod\ 4$ for $i \in \{0, 1, 2, 3\}$. If you check, $f(i + j \ mod\ 4) = 2(i + j) \ mod\ 4$ and this is the same as $f(i) + f(j) = (2i \ mod\ 4) + (2j \ mod\ 4) \ mod\ 4 = 2(i + j) \ mod\ 4$.

I started babbling about how these functions are 1-1 and onto. I didn't want to spend too much class time defining these concepts, but they are important and come up everywhere in mathematics. When f maps G_1 to G_2 then G_1 is the domain and G_2 is called the co-domain. I like to use the language that an element $x \in G_1$ is 'sent to' an element $f(x)$ in G_2 so that I can say that intuitively 1-1 means that a function 'sends every element in the domain to a different element in the co-domain.' More precisely,

Definition 1. A function f that maps G_1 to G_2 is $1-1$ if $x, y \in G_1$ and $x \neq y$, then $f(x) \neq f(y)$.

Then I also like to say that an element y in the codomain is 'hit' if there is some x such that $f(x) = y$. A function is onto means that every element in the codomain is 'hit.' More precisely,

Definition 2. A function f that maps G_1 to G_2 is onto if for every $y \in G_2$, there is an element x in G_1 such that $f(x) = y$.

Example 3 is $1-1$, but not onto. Example 4 is onto, but not $1-1$. Example 5 is neither $1-1$ nor onto. Example 2 is both $1-1$ and onto (an isomorphism, bijection).

I then talked about the group of permutations of n and cycle notation. A permutation σ is a bijection from the numbers $\{1, 2, \ldots, n\}$ to the numbers $\{1, 2, \ldots, n\}$. We will represent σ in cycle notation, that is write it as

$$\sigma = (i_1, i_2, \ldots, i_{c_1})(j_1, j_2, \ldots, j_{c_2}) \cdots (\ell_1, \ell_2, \ldots, \ell_{c_r})$$

where the integers $\{1, 2, \ldots, n\}$ appear exactly once in the permutation. This notation means

$\sigma(i_k) = i_{k+1}$ for $1 \leq k < c_1$ and $\sigma(i_{c_1}) = i_1$
$\sigma(j_k) = j_{k+1}$ for $1 \leq k < c_2$ and $\sigma(j_{c_2}) = j_1$
$\vdots$
$\sigma(\ell_k) = \ell_{k+1}$ for $1 \leq k < c_r$ and $\sigma(\ell_{c_1}) = \ell_1$

The set if permutations of n represented this way with composition of permutations $\sigma \circ \tau$ is the permutation where $\sigma \circ \tau(i) = \sigma(\tau(i))$. I then tried to do an example with $n = 3$, but realized that the example is too small so I tried a larger example so that it is clear what I meant and how. Take $\sigma = (1, 3, 4)(2, 5, 6)(7)$ and $\tau = (1)(2, 3, 5)(4)(6, 7)$. The permutation σ should be read as "1 is sent to 3, 3 is sent to 4, 4 is sent to 1, 2 is sent to 5, 5 is sent to 6, 6 is sent to 2, 7 is sent to 7." or just $\sigma(1) = 3$, $\sigma(3) = 4$, $\sigma(4) = 1$, $\sigma(2) = 5$, $\sigma(5) = 6$, $\sigma(6) = 2$, $\sigma(7) = 7$. Similarly, the permutation τ should be read as $\tau(1) = 1$, $\tau(2) = 3$, $\tau(3) = 5$, $\tau(5) = 2$, $\tau(4) = 4$, $\tau(6) = 7$, $\tau(7) = 6$.

Let me try to indicate how we give the notation for $\sigma \circ \tau$, we start with by asking where 1 is sent (we can start with any integer, but this is a good place to start). In step 1, we have

$$\sigma \circ \tau = (1, \ldots$$

Then $\tau(1) = 1$ and $\sigma(\tau(1)) = \sigma(1) = 3$. Stated in words, τ sends 1 to 1 and σ sends it to 3. We record,

$$\sigma \circ \tau = (1, 3, \ldots$$

$\sigma(\tau(3)) = \sigma(5) = 6$ or in words 3 is sent to 5 under τ and 5 is sent to 6 under σ.

$$\sigma \circ \tau = (1, 3, 6, \ldots$$

$\sigma(\tau(6)) = \sigma(7) = 7$. In words again, 6 is sent to 7 under τ and 7 is sent to 7 under σ.

$$\sigma \circ \tau = (1, 3, 6, 7, \ldots$$

$\sigma(\tau(7)) = \sigma(6) = 2$. In other words, 7 is sent to 6 by τ and 6 is sent to 2 by σ.

$$\sigma \circ \tau = (1, 3, 6, 7, 2, \ldots$$

$\sigma(\tau(2)) = \sigma(3) = 4$.

$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4, \ldots$$

$\sigma(\tau(4)) = \sigma(4) = 1$.

$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4) \ldots$$

So far we have explained where everything except 5 is sent, so we add another cycle beginning with 5 (we would normally take any of the remaining elements that are not in a cycle yet).

$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4)(5, \ldots$$

$\sigma(\tau(5)) = \sigma(2) = 5$. That is τ sends 5 to 2 and σ sends 2 to 5. For this reason, we then close the parenthesis to indicate that 5 is sent to 5 under $\sigma \circ \tau$.

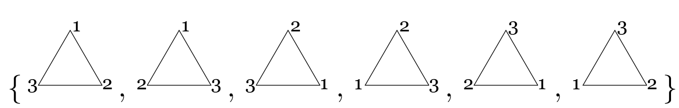
$$\sigma \circ \tau = (1, 3, 6, 7, 2, 4)(5)$$

Since all of the integers 1 through 7 appear once in this expression we know that we are done.

Many of the examples we have considred above are not just groups, but the groups are motions of a square or a triangle. In other words they can be thought of as acting on a set of objects. We have a notion of this that I will introduce here.

Definition 3. A group action on a set X is a map $\bullet : G \times X \to X$ such that $e \bullet x = x$ for all $x \in X$ and $g \bullet (h \bullet x) = (gh) \bullet x$ for all $g, h \in G$ and $x \in X$.

For example, the set of motions of a triangle acts on the set



but you can also think of the motions as acting on just the vertices themselves $\{1, 2, 3\}$. For example $R_{120}(1) = 3$ (remember that we said that $R_{120} = (132)$). These groups are not really big enough to give a good clear example so I will wait until I have the group of the motions of a cube to give more examples.

Definition 4. The orbit of an element $x \in X$ is the set (it is a subset of X)

$$O_x = \{g \bullet x : g \in G\}$$

Definition 5. The stabilizer of an element $x \in X$ is a set (it is a subset of G)

$$Stab(x) = \{g \in G : g \bullet x = x\}$$

I will give some examples of the orbits and stabilizers when we have some better group actions. But for the moment consider the action of the group of motions of the square on the set of diagonals $\boxbslash$, $\boxslash$. Then $e \bullet \boxbslash = R_{180} \bullet \boxbslash = F_{D_1} \bullet \boxbslash = F_{D_2} \bullet \boxbslash = \boxbslash$ while $R_{90} \bullet \boxbslash = R_{270} \bullet \boxbslash = F_V \bullet \boxbslash = F_H \bullet \boxbslash = \boxslash$ This defines a group action on the diagonals of the square (you will also need to figure out the action of the elements on $\boxslash$, but these are enough to define the action.

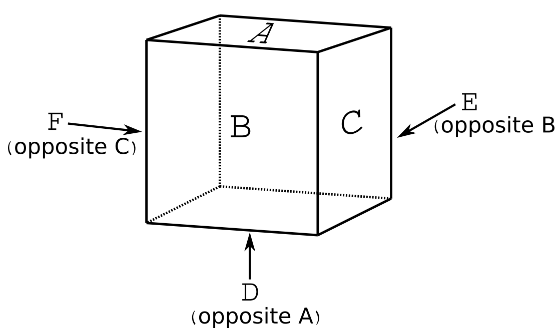
The orbit of $\boxbslash$ is $O_{\boxbslash} = \{\boxbslash, \boxslash\}$. The stabilizer of $\boxbslash$ is $Stab(\boxbslash) = \{e, R_{180}, F_V, F_H\}$.

Next I decided to say that we were ready to determine the group of motions of a cube. We can tell how many motions of a cube there are by a counting argument. If we label the 6 faces, then there are 6 ways of choosing which face will be up and then 4 ways of choosing which face will be in front. Therefore every motion of the cube is determined by these two steps so the number of motions of a cube is equal to $6 \cdot 4 = 24$.

Remark 6. A cube has 6 faces, 8 vertices and 12 edges. The number $24 = 4!$ which is equal to the number of permutations of $\{1, 2, 3, 4\}$. Here is a good question: is it possible to recognize the motions of the cube as the permutations of 4 things on the cube so that it is clear that these two groups are the same (isomorphic)?

So I gave you access to a cube to follow along because the cube I was working with was not big enough to see from a distance. It is a good idea in the following discussion to have a cube on hand to be able to better visualize what I am trying to communicate.

Take a cube and label the faces with the letters A, B, C, D, E, F.



One motion of the cube fixes all faces and is the identity of the group.

$$e = (A)(B)(C)(D)(E)(F)$$

Then it is possible to fix the face A and D and rotate the cube while keeping that face fixed. There are three rotations (besides the one where all faces are fixed.

$$(A)(D)(BCEF)$$
$$(A)(D)(BE)(CF)$$
$$(A)(D)(BFEC)$$

But we can also fix B and E and rotate around those faces

$$(B)(E)(ACDF)$$
$$(B)(E)(AD)(CF)$$
$$(B)(E)(AFDC)$$

and we can fix C and F and rotate around those faces

$$(C)(F)(ABDE)$$
$$(C)(F)(AD)(BE)$$
$$(C)(F)(AEDB)$$

Now we have found 10 motions of the cube and expressed them in terms of their action on the faces, but that is less than half since we are looking for 24. Now look at the top face labeled with A and pick one of the four edges that adjoins the faces B, C, E or F and then there is an edge which is furthest away from that edge. You can flip the cube across those two edges leaving them fixed and all the others edges are permuted. These correspond to the motions

$$(AB)(DE)(CF)$$
$$(AC)(DF)(BE)$$
$$(AE)(DB)(CF)$$
$$(AF)(DC)(BE)$$

There are two more of these kinds of flips where we flip across the edge which adjoins B and C and the corresponding edge between E and F and the edge adjoining B and F and E and C which are

$$(BC)(EF)(AD)$$
$$(BF)(EC)(AD)$$

Great, now we have 16 of the 24 motions of the cube. We need 24 in total. Exercise: find the other 8. Hint: look at the motions which fix diagonals across opposite corners. We haven't yet looked at those.

NOTES ON NOV 8, 2012

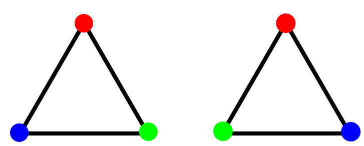
MIKE ZABROCKI

I started off with an exercise that I asked everyone to do while I put the group of motions of the cube on the board. How many ways are there of coloring the vertices of a triangle with the colors R, G, B such that two colorings are considered to be the same if one can be obtained from another by the action of an element of the group?

group	allowing repeated colors	exactly one of each color
$\{e\}$	3^3	6
$\{e, R_{120}, R_{240}\}$	11	2
$\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$	10	1

The first row we didn't really need any discussion to figure out. If we allow repeated colors and no two colorings are equivalent, then there are three choices for each vertex and hence 3^3 colorings. If we are allowed to use each color once then there are $3! = 6$ colorings.

If all three colors are different then under the group of rotations of the triangle the two colorings

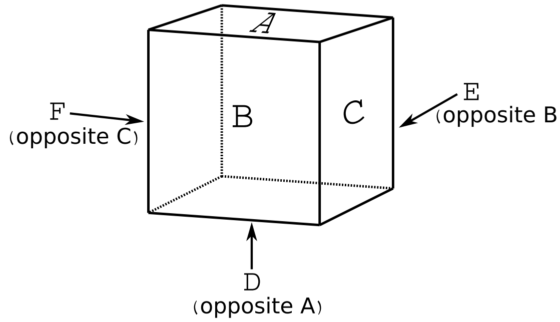


are both different, but all other colorings are equivalent to one of these two. If we are allowed to use each color more than once then there are three colorings where the three vertices are the same color, there are 6 colorings that where two of the first color and one of a second color, and the two colorings that are shown above. Therefore there are 11 different colorings in total.

If we consider the group of rotations and flips, then the two colorings that are shown above suddenly equivalent but the others are still unique and hence there is just one less coloring under this group (and so there are 10).

The reason I wanted to do this exercise is that we are working our way towards coming up with a formula for counting these things (we can do small examples like this by hand, but larger examples are difficult to enumerate).

Take a cube and label the faces with the letters A, B, C, D, E, F.



Last time I listed 16 elements of the group and I asked you to find the remaining 8. Those missing 8 were are the ones that are listed last below.

	$(C)(F)(ABDE)$	$(ABC)(DEF)$
$e = (A)(B)(C)(D)(E)(F)$	$(C)(F)(AD)(BE)$	$(ACB)(DFE)$
$(A)(D)(BCEF)$	$(C)(F)(AEDB)$	$(ABF)(DEC)$
$(A)(D)(BE)(CF)$	$(AB)(DE)(CF)$	$(AFB)(DCE)$
$(A)(D)(BFEC)$	$(AC)(DF)(BE)$	$(AEC)(DBF)$
$(B)(E)(ACDF)$	$(AE)(DB)(CF)$	$(ACE)(DFB)$
$(B)(E)(AD)(CF)$	$(AF)(DC)(BE)$	$(AEF)(DBC)$
$(B)(E)(AFDC)$	$(BC)(EF)(AD)$	$(AFE)(DCB)$
	$(BF)(EC)(AD)$	

We know that we have them all because we had a combinatorial argument that explained why there must be 24 motions of the group. But why is this a group?

(1) If you do a motion of the cube, and then you do a second motion of the cube you will have completed some motion of the cube. Therefore the set of motions is closed under the operation of composition.
(2) Doing nothing to the cube is the identity element of the group and is represented by the element $(A)(B)(C)(D)(E)(F)$.
(3) If you move a cube and then it is always possible to undo that movement and that will still be a motion of the cube. Therefore the inverse of every motion exists.

These three conditions are all that are required to for our set of motions with the operation of composition to form a group.

EXERCISE: Let $D1 =$ the diagonal between the vertex at corner of the faces ABC and the vertex at corner of the faces DEF, $D2 =$ the diagonal between the vertex at corner of the faces ABF and the vertex at corner of the faces DEC, $D3 =$ the diagonal between the vertex at corner of the faces AEC and the vertex at corner of the faces DBF, $D4 =$ the diagonal between the vertex at corner of the faces AEF and the vertex at corner of the faces DBC. Rewrite each of the motions of the cube as a permutation of the elements of $\{D1, D2, D3, D4\}$.

I wanted to use this group then to demonstrate the definitions of the orbit and stabilizer of an element x. I made a table and calculated an example of the group of motions of a cube acting on some sets of elements. I made a table and we calculated the orbit and the stabilizer of a few different elements. These motions of the cube act on the cube but you can think of these motions acting on the faces, edges, vertices or even combinations of these things.

object x	orbit O_x	stabilizer $stab(x)$
the face A	$\{A, B, C, D, E, F\}$	$\{(A)(B)(C)(D)(E)(F),$ $(A)(D)(BCEF),$ $(A)(D)(BE)(CF),$ $(A)(D)(BFEC)\}$
the edge adjoining AB	the edges adjoining $\{AB, AC$ $AE, AF, BC, BD, BF,$ $CD, CE, DE, DF, EF\}$	$\{(A)(B)(C)(D)(E)(F),$ $(AB)(DE)(CF)\}$
the vertex at the corner of ABC	vertices at $\{ABC, ABF, AEC,$ $AEF, DBC, DEC, DBF, DEF\}$	$\{(A)(B)(C)(D)(E)(F),$ $(ABC)(DEF),$ $(ACB)(DFE)\}$

The thing you should notice is the relationship between the number of elements in the orbit and the number of elements in the stabilizer. When the orbit has 6 elements, the stabilizer has 4. When the orbit has 12 elements, the stabilizer has 2 elements. When the orbit has 8 elements, the stabilizer has 3. This should lead you to make the following conjecture:

Theorem 1. *(the orbit stabilizer theorem) The product of the number of elements in the orbit and the number of elements in the stabilizer is equal to the number of elements in the group, or in equation form*

$$|O_x| \cdot |Stab(x)| = |G| \ .$$

In order to show why this is true we need to develop a few results. The first is that the set $Stab(x)$ is a subgroup of G, that is it is a subset of G and is itself a group.

Lemma 2. *The set $Stab(x)$ is a group.*

Proof. We need to show that $Stab(x)$ is (1) closed under multiplication, that it (2) contains the identity and that it (3) contains the inverse of every element that is in the set. This all follows from the definition of group and group action. Let the multiplication in the group be denoted by $\circ$ and the action on the element x by $\bullet$.

(1) if $f, g \in Stab(x)$, then $f \bullet x = g \bullet x = x$ (by definition), hence $(f \circ g) \bullet x = f \bullet (g \bullet x) = f \bullet x = x$, therefore $f \circ g \in Stab(x)$.
(2) by the definition of group action $e \bullet x = x$, hence $e \in Stab(x)$.
(3) if $f \in Stab(x)$, then $f^{-1} \in G$ and $f^{-1} \bullet x = f^{-1} \bullet (f \bullet x) = (f^{-1} \circ f) \bullet x = e \bullet x = x$, hence $f^{-1} \in Stab(x)$.

□

In order to talk about some results we will need next I need the notion of a relation. I cover relations in Math 1200. A relation on a set X is a set of pairs $a \sim b$ where $a, b \in X$. Example of relations are things like a is greater than b, a is better than b, a is equal to b, a is taller than b, a older than b, a and b are second cousins (the set of things that a and b might be from differ wildly in those examples but you can guess from context what a and b might be, but you can have a relation on any set).

Example 1: On the set of integers $a \sim b$ if $a = b$
Example 2: On the set $\{1, 2, 3\}$, $1 \sim 2$, $2 \sim 3$
Example 3: on the set of integers $a < b$
Example 4: on the set of integers $a \leq b$

Definition 3. A relation $\sim$ on a set X is said to be reflexive if $a \sim a$ for all $a \in X$.

Definition 4. A relation $\sim$ on a set X is said to be symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in X$.

Definition 5. A relation $\sim$ on a set X is said to be transitive if $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in X$.

Any relation can have any one or none of these properties Examples 1 and 4 are reflexive, only Example 1 is symmetric, Example 1, 3 and 4 are transitive and Example 2 is not reflexive, symmetric nor transitive.

Definition 6. A relation $\sim$ on a set X is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Only Example 1 is an equivalence relations, but there are other equivalence relations which are not equals (e.g. two colorings are equivalent if one can be obtained from the other by a motion in the group G).

Proposition 7. *Let H be a subgroup of the group G. Define the relation on the group G so that for $a, b \in G$, $a \equiv b$ if there is an $h \in H$ such that $a = bh$. $\equiv$ is an equivalence relation.*

This proposition relies on the properties of the fact that H and G are groups to show that it is reflexive, symmetric and transitive. This relation on G depends on H so sometimes the relation is denoted $\equiv_H$ or $a \equiv b \ (mod\ H)$ to indicate what the subgroup is, but if it is clear then the reference to H is usually dropped.

Proof. We need to show that this relation is (1) reflexive, (2) symmetric and (3) transitive.

(1) Since $e \in H$ so $a = a \circ e$ implies $a \equiv a$.
(2) If $a \equiv b$, then $a = b \circ h$ so $b = a \circ h^{-1}$ and since H is a subgroup $h^{-1} \in H$, so $b \equiv a$.
(3) If $a \equiv b$ and $b \equiv c$, then $a = b \circ h$ and $b = c \circ h'$ so $a = (c \circ h') \circ h = c \circ (h \circ h')$ and so $a \equiv c$.

Therefore $\equiv$ is an equivalence relation. □

The reason I wanted to introduce equivalence relations (and this one in particular) is that an equivalence relation on a set of elements partitions that set of elements. Think, for example, the colorings of triangles that we started this class with. A coloring is equivalent to another one if there is a motion of the group that takes one to the other (an example of an equivalence relation). Now when I listed the number of colorings, I was saying that every coloring is equivalent to one of these that are counted in that first table. We partitioned the set into things that are equivalent to each other. We need to justify what we did in that example 'every element is equivalent to one of these representatives.'

Let a, b, c, d be elements of some set and let $c \sim d$ be an equivalence relation on that set. Define the equivalence class of a to be

$$C_a = \text{ the set of elements related to } a \ = \{c : c \sim a\}$$

To say that an element d which is related to a and related to b is to say that $d \in C_a$ and $d \in C_b$. To say that 'every element is equivalent to one of these representatives' I mean that I want to show that $C_a = C_b$.

Proposition 8. *Every two equivalence classes C_a and C_b either have no elements in common or they are equal.*

NOTES ON NOV 13, 2012

MIKE ZABROCKI

In the previous class I had set up that we wanted to show the orbit-stablizer theorem. That is, we want to show,

$$|O_x| \cdot |Stab(x)| = |G| \ .$$

Recall a couple of statements we have so far:

- if G acts on x, then $Stab(x) = \{g : g \bullet x = x\}$ is a subgroup of G
- For any subgroup H, $\equiv_H$ is an equivalence relation on G

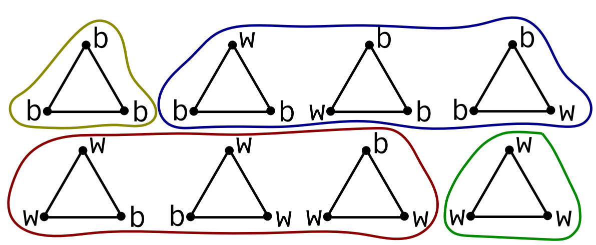
From last time I introduced the vocabulary and notation: orbit O_x, stabilizer $Stab(x)$, subgroup, reflexive relation, symmetric relation, transitive relation, equivalence relation, equivalence class C_a. I realize that this a vocabulary heavy period of the course, but these concepts are given names because they come up over and over in group theory.

I set up two other statements that I need to use to justify the orbit-stabilizer theorem.

(1) If C_a and C_b are the equivalence classes of a and b under some equivalence relation, then either C_a and C_b have no elements in common or the two sets C_a and C_b are equal.
(2) In particular, with the equivalence classes of the relation $\equiv_H$, all equivalence classes have the same number of elements.

Since I want to justify these statements, let me give a few examples of equivalence relations and equivalence classes so that we can convince ourselves that at least these statements are true on some small examples. Also I want to convince you that there is something important to show here and that statement number (2) is not always true.

Consider the set of colorings of the vertices of a triangle with B and W such that two colorings are equivalent if one can be obtained from another by rotation. That is, $coloring_1 \sim coloring_2$ if there is a $g \in \{e, R_{120}, R_{240}\}$ such that $g \bullet coloring_1 = coloring_2$. Lets try coloring the vertices of a triangle with b and w such that two colorings are distinct if they are the same under a rotation of the shape. Lets draw them:



I have drawn a loop around the colorings which are equivalent to each other under rotation. These groupings are called the orbits under the action of the group. We have a goal of counting the number of orbits in our set of colorings. We can see that there are two orbits with one element each and two orbits with 3 elements each.

Lets consider one more example, but this time with our equivalence relation $\equiv_H$. This time take our group G to be $G = \{e, R_{120}, R_{240}, F_1, F_2, F_3\}$ and the subgroup $H = \{e, R_{120}, R_{240}\}$ is used to define the equivalence relation $g_1 \equiv_H g_2$ if there exists an $h \in H$ such that $g_1 h = g_2$.

I note that in particular we have that $e \circ R_{120} = R_{120}$ so we know that $e \equiv_H R_{120}$. We also have that $R_{120} \circ R_{120} = R_{240}$ then $R_{120} \equiv_H R_{240}$. This also implies that $e \equiv_H R_{240}$ because we know that this relation is transitive. It is the case that the only elements of G which are equivalent to e are the elements of H because H is closed.

So then if we look at F_1, we find that $F_1 \circ R_{120} = (1)(23) \circ (132) = (12)(3) = F_3$. We also calculate that $F_3 \circ R_{120} = F_2$ and hence $C_{F_1} = \{F_1, F_2, F_3\}$.

One thing that is different about this example than the example with colorings of triangles is that there are two equivalence classes and they are both of the same size. It turns out with the equivalence relation $\equiv_H$ the equivalence classes are all the same size. Its hard to tell from a small example like this that the property continues.

Lemma 1. *Let $\sim$ be an equivalence relation and set $C_a = \{x : x \sim a\}$ (the set of things which are equivalent to a). If C_a and C_b have one element in common, then the sets are equal.*

In order to show why this is true, we need to show two sets are equal. The usual method for doing this is to show that $C_a \subseteq C_b$ and the reverse inclusion.

Proof. Say that C_a and C_b have an element d in common. That is, $d \sim a$ and $d \sim b$. Since $\sim$ is symmetric, $a \sim d$. Since $\sim$ is transitive and $a \sim d$ and $d \sim b$, then $a \sim b$. Let f be an element in C_a. By definition of C_a, $f \sim a$ and since $a \sim b$, then $f \sim b$, hence $f \sim b$ and $f \in C_b$. □

I then recalled that the statement $A \Rightarrow B$ is logically equivalent to *not A or B* (I even went so far as to draw the truth table for both of them to verify this). This means that the sentence

If C_a and C_b have an element in common, then $C_a = C_b$.

is equivalent to

Either C_a and C_b don't have an element in common, or $C_a = C_b$.

And this last statement is the same as (1).

Now since in an equivalence relation, every element is equivalent to some element because (at the very least) it is equivalent to itself. Hence every element is in some equivalence class and these equivalence classes are all disjoint so they form a partition of the set of elements.

Remark 2. You should note that a partition of a set also determines an equivalence relation by declaring that $a \sim b$ is if a and b are in the same part of the set partition. Therefore the number of set partitions on an n element set (the Bell numbers B_n given by the sequence $1, 1, 2, 5, 15, 52, \ldots$) is equal to the number of distinct equivalence relations on the set $\{1, 2, 3, \ldots, n\}$.

I also showed that every equivalence class of the equivalence relation $\equiv_H$ has the same number of elements. This is a special property and holds because H and G are groups. Let me rewrite the number of elements in the equivalence class of G. They are

$$C_g = \{g' : g'h = g \text{ for some } h \in H\} = \{g' : g' = gh^{-1} \text{ for some } h \in H\} = \{gh : h \in H\}$$

The reason that the third equality is true is because H is a group so running over all $h^{-1} \in H$ is the same as running over all $h \in H$. I then defined new notation for the set on the right hand side of the equality

$$gH := \{gh : h \in H\}\ .$$

These sets are called the (left) cosets of H.

What I want to show is that the equivalence classes of $\equiv_H$ are all the same size as the set H. Since the equivalence classes of $\equiv_H$ are all of the form gH for some g, then all I need to do is show that gH has the same size as H no matter what $g \in G$ is. In order to show that gH has the same size as H I need to find a bijection between the elements of H and the elements of gH.

Lemma 3. *The equivalence classes of $\equiv_H$ which partition the set G all have the same size. Since these equivalence classes are of the form gH for some g, they all have the same size as the subgroup $H = eH$.*

Proof. I want to define a bijection between H and gH. To do this I define the map ϕ_g which maps subsets $S \subseteq G$ to another subset $\phi_g(S) = \{gk : \text{ for } k \in S\}$. In particular, $\phi_g(H) = gH$. Because groups have so much structure, it will be the case that $\phi_g(H)$ and H have the same number of elements because ϕ_g is a bijection. How do we know? $\phi_{g^{-1}}(\phi_g(H)) = \phi_{g^{-1}}(gH) = \phi_{g^{-1}}(\{gh : h \in H\}) = \{g^{-1}gh : h \in H\} = H$ so there is a left inverse. The same calculation also shows that $\phi_g(\phi_{g^{-1}}(H)) = H$ so there is a right inverse, this means that ϕ_g is a bijection between H and gH and hence they have the same number of elements. $\square$

I claim now that we have enough facts about sets, orbits, stabilizers, equivalence classes, groups, etc. to allow us to justify the orbit stabilizer theorem. We know that the stabilizer is a subgroup of G, therefore the equivalence relation $\equiv_{Stab(x)}$ partitions G and every equivalence class has the same number of elements. Conclusion:

$$|G| = |Stab(x)| \cdot \text{ the number of different equivalence classes of } \equiv_{Stab(x)}$$

But we want to show that $|G| = |Stab(x)| \cdot |O_x|$ so we just need to show that the number of different equivalence classes $= |O_x|$. In general, to show that two sets of objects have the same number of elements you show that there is a bijection between them. In this case we are looking for a bijection between the set of equivalence classes of $\equiv_{Stab(x)}$ and the elements of O_x. Remember that the equivalence classes of $\equiv_{Stab(x)}$ are the sets $gStab(x) = \{gh : h \in Stab(x)\}$.

Lemma 4. *the number of different equivalence classes of $\equiv_{Stab(x)}$ is equal to the number of elements in O_x.*

Proof. What we will do is define a bijection between the equivalence classes of $\equiv_{Stab(x)}$ (the cosets $gStab(x)$) and the elements of O_x. For a coset $g'Stab(x)$ of G, let $\psi(g'Stab(x))$ be defined as taking an element of $g \in g'Stab(x)$ and the result is $g \bullet x$. This maps a set $g'Stab(x)$ to an element of O_x. We need to show the following

(1) First, ψ must be well defined because there was some sort of arbitrary step that we did when we we took 'an element' from $g'Stab(x)$. How do we know that we get the same result each time?
(2) Second, we need to show that if you take two cosets $g'Stab(x)$ and $g''Stab(x)$ and if we find that $\phi(g'Stab(x)) = \phi(g''Stab(x))$, then $g'Stab(x) = g''Stab(x)$ (that is we need to know that this map is 1-1).
(3) Finally, we need to know that every element in the orbit of x, $y \in O_x$, there some coset $g'Stab(x)$ such that $\psi(g'Stab(x)) = y$ (that is that this map is onto).

If we have all three of these properties then we know that ψ is a well defined bijection between the cosets of $Stab(x)$ and the elements of O_x.

The first statement is true because if g_1 and g_2 are in $g'Stab(x)$, then $g_1 = g'h_1$ and $g_2 = g'h_2$ where $h_1 \bullet x = h_2 \bullet x = x$ so then

$$g_1 \bullet x = (g'h_1) \bullet x = g' \bullet (h_1 \bullet x) = g' \bullet x = g' \bullet (h_2 \bullet x) = (g'h_2) \bullet x = g_2 \bullet x \ .$$

This says that no matter which elements we take from $g'Stab(x)$ that we get the same value $g' \bullet x$.

The second statement is true because if $\phi(gStab(x)) = \phi(g'Stab(x))$ then $g \bullet x = g' \bullet x$ (because $g \in gStab(x)$ and $g' \in g'Stab(x)$ so by part (1) we know we can take these in particular) so

$$x = (g^{-1}g) \bullet x = g^{-1} \bullet (g \bullet x) = g^{-1} \bullet (g' \bullet x) = (g^{-1}g') \bullet x \ .$$

Therefore $g^{-1}g' \in Stab(x)$ and so $Stab(x) = \{g^{-1}g'h : h \in Stab(x)\}$ and

$$gStab(x) = \{gh : h \in Stab(x)\} = \{gg^{-1}g'h : h \in Stab(x)\} = \{g'h : h \in Stab(x)\} = g'Stab(x) \ .$$

The third statement is true because if $y \in O_x$ then there is some element $g \in G$ such that $y = gx$ (because that is what it means for y to be in the orbit of x). But then, $\psi(gStab(x)) = g \bullet x = y$. $\square$

Remark 5. The number of different equivalence classes of $\equiv_H$ (or the number of different left cosets of a subgroup H) is called the index of H in G. I wanted to avoid introducing one more name, definition, notation in this case because we don't really use it, but the name occurs frequently in group theory.

NOTES ON NOV 15, 2012

MIKE ZABROCKI

In our last episode I showed you that,

$$|O_x| \cdot |Stab(x)| = |G| \ .$$

We are just one short calculation away from the result that we have been building up to for a while. Since I want to show it off, I am going to state it, give a bunch of examples (actually I will revisit some of the examples that we looked at already) and then I will justify why the formula is correct.

Theorem 1. *(Burnside's Lemma) Let G be a group which acts on a set of elements X,*

$$\text{The number of orbits when } G \text{ acts on } X = \frac{1}{|G|}\sum_{g\in G} \# \text{ of elements fixed by } g \ .$$

The reason that I say that we have now reached the point where we have given a formula for the examples that we have been discussing for the last couple of weeks is when we talk about colorings being equal we mean that they are in the same orbit. When we talk about different colorings, we are talking about two colorings being in different orbits under the action of G. So when we want to know how many different colorings there are, we want to know how many different orbits there are under the action of G and Burnside's Lemma is a formula for exactly that.

Remember on November 8 we figured out (by more or less writing down all possible colorings) the number of colorings of the vertices of a triangle under the action of three different groups, $\{e\}$, $\{e, R_{120}, R_{240}\}$ and $\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$. We arrived at the following table (there was a second column of this table but we will concentrate on just the first column. As an exercise figure out how the formula applies to the second column):

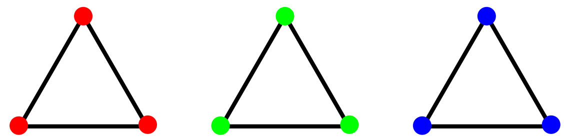
group	allowing repeated colors
$\{e\}$	3^3
$\{e, R_{120}, R_{240}\}$	11
$\{e, R_{120}, R_{240}, F_1, F_2, F_3\}$	10

For the first row of this table it says that because the identity fixes all 3^3 possible colorings of the triangle that the number that are different under the group $\{e\}$ is equal to

$$\frac{1}{|\{e\}|}3^3 = \frac{1}{1} \cdot 27 = 27 \ .$$

This example isn't very enlightening. But lets consider the other two.

When R_{120} and R_{240} act on the triangle, the only colorings that are fixed are those where all three vertices are colored exactly the same, that is:

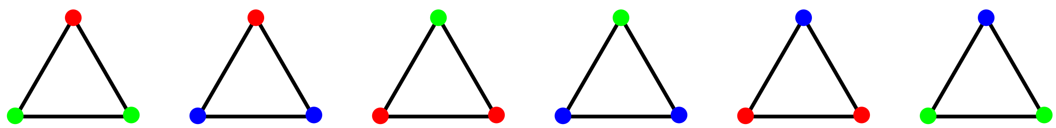


That means that the total number of different colorings under the group $\{e, R_{120}, R_{240}\}$ is equal to

$$\frac{1}{3}(3^3 + 3 + 3) = \frac{1}{3} \cdot 33 = 11$$

and this agrees with the table that we had calculated before.

If we look under the action of F_1, in addition to the three pictured colorings above, there are 6 others:



So in total, there are 9 colorings which are fixed by F_1. Similarly there are 9 which are fixed by F_2 and 9 which are fixed by F_3. Burnside's Lemma then tells us that the total number of different colorings by the action of this group is equal to

$$\frac{1}{6}(27 + 3 + 3 + 9 + 9 + 9) = \frac{1}{6} \cdot 60 = 10 \ .$$

Recall that the group elements have the following cycle structure

$$e = (1)(2)(3), R_{120} = (132), R_{240} = (123), F_1 = (1)(23), F_2 = (2)(13), F_3 = (3)(12) \ .$$

Unless there are other restrictions on the colors the number of elements in

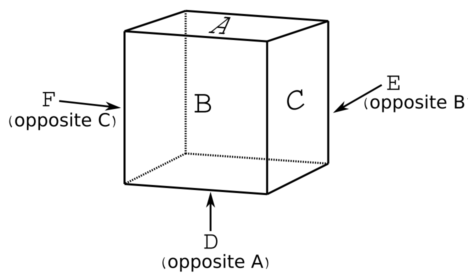
$$Fix(g) = (\# \text{ number of colors})^{(\# \text{ of cycles in } g)}$$

In particular we see $Fix(F_1) = Fix(F_2) = Fix(F_3) = 3^2$, $Fix(R_{120}) = Fix(R_{240}) = 3$ and $Fix(e) = 3^3$.

What is kind of cool about this formula is that just by looking at the expression, it is not clear that the order of the group in the denominator is going to cancel with the sum over the elements which are fixed by the group elements, but in the end it does. In fact, we can use this as a (weak) check that we haven't made any mistakes in our calculations by ensuring that the denominator does cancel with the numerator. If you get a rational number for the number of orbits, check again.

The reason this formula is useful, is that in general there are not that there are generally more colorings than there are group elements and another reason is that it is usually not that difficult to figure out how many elements are fixed by any particular group element g. Moreover, a lot of group elements have the same number of elements of x which are fixed by G.

Let me try to count the number of ways of coloring the faces of a cube with colors black and white such that two coloring are the same if one can be obtained from another by a motion of the cube. Fortunately we have already calculated the group of the motions of the cube. Label the faces of the cube with the letters A through F as in the following diagram.



Recall that the group of motions of the cube consisted of the following elements.

	$(C)(F)(ABDE)$	$(ABC)(DEF)$
$e = (A)(B)(C)(D)(E)(F)$	$(C)(F)(AD)(BE)$	$(ACB)(DFE)$
$(A)(D)(BCEF)$	$(C)(F)(AEDB)$	$(ABF)(DEC)$
$(A)(D)(BE)(CF)$	$(AB)(DE)(CF)$	$(AFB)(DCE)$
$(A)(D)(BFEC)$	$(AC)(DF)(BE)$	$(AEC)(DBF)$
$(B)(E)(ACDF)$	$(AE)(DB)(CF)$	$(ACE)(DFB)$
$(B)(E)(AD)(CF)$	$(AF)(DC)(BE)$	$(AEF)(DBC)$
$(B)(E)(AFDC)$	$(BC)(EF)(AD)$	$(AFE)(DCB)$
	$(BF)(EC)(AD)$	

- The identity $(A)(B)(C)(D)(E)(F)$ fixes all colorings and since we can choose b or w for each face, there are 2^6 colorings which are fixed by the identity.
- Say that we fix two faces then there are two types of permutation, those that rotate by ± 90 degrees (e.g. $(A)(D)(BCEF)$ or $(A)(D)(BFCE)$) and those that rotate by 180 degrees . The ones that rotate by $\pm 90°$ fix all colorings where all the 4 faces which move are the same color. There are two choices for the 4 faces and 2 choices for each of the two fixed faces. In total there are 2^3 colorings which are fixed by rotations by $\pm 90°$.
- The ones that rotate by $180°$ (e.g. $(A)(D)(BE)(CF)$) fix all colorings where the opposite faces that exchange are the same color. We have 2 choices for each of the two fixed faces and 2 choices for the two pairs of faces which exchange. We can read from the cycle structure of these permutations that there are 4 cycles and as long as each cycle has the same color and so in total there are 2^4 ways of coloring those faces.

- The permutations which fix an edge (e.g. $(AB)(DE)(CF)$) then there are three pairs of faces which are exchanged and they must be colored the same color and so there are 2^3 colorings which are fixed by these permutations.
- The permutations which fix a vertex and rotate by $\pm 120°$ (e.g. $(ABC)(DEF)$) must have the three faces which are all clustered around the vertex that is being rotated around all the same color therefore there are 2^2 colorings.

Look at the list of group elements above. We have:

- one identity element $(A)(B)(C)(D)(E)(F)$
- six rotations about two fixed faces by $\pm 90°$ (e.g. $(A)(D)(BCEF)$)
- three rotations about two fixed faces by $180°$ (e.g. $(A)(D)(BE)(CF)$)
- six flips about an edge (e.g. $(AB)(DE)(CF)$)
- eight rotations about a vertex by $\pm 120°$ (e.g. $(ABC)(DEF)$)

Burnside's Lemma then says that the number of colorings of a cube with black and white edges is equal to

$$\frac{1}{24}(2^6 + 6 \cdot 2^3 + 3 \cdot 2^4 + 6 \cdot 2^3 + 8 \cdot 2^2) = \frac{1}{24} \cdot 240 = 10 \ .$$

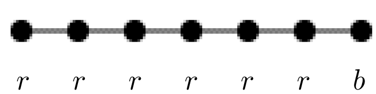
Now look back at your notes from October 30 and that was when we first started talking about colorings of the cube. I then said that the generating function for the number of colorings of the cube with black and white faces is:

$$B^0W^6 + B^1W^5 + 2B^2W^4 + 2B^3W^3 + 2B^4W^2 + B^5W^1 + B^6W^0 \tag{1}$$

I will show you by the end of the class how we can give a formula for this generating function but if you add up all of the coefficients (the total number of colorings) it is $1 + 1 + 2 + 2 + 2 + 1 + 1 = 10$.

So I asked you on the homework to count the number of ways of coloring the vertices of the trees with 7 vertices using k colors such that two colorings are equal if one can be transformed to another by sending vertices to vertices and edges to edges. I thought I would show a single example of how I would like you to apply this formula to answer this question. Consider the colorings of the following graph.

Now notice that the group consisting of the identity and the motion which flips the tree backwards are the only two elements which preserve the tree structure. I want to count colorings where (for instance) the following two colorings are the same:



$b \quad r \quad r \quad r \quad r \quad r \quad r$

The way that we will go about doing this is to first label the vertices of the tree with the numbers 1 through 7 so that we can refer to them.

1 2 3 4 5 6 7

Then the two group elements which act on this tree are $e = (1)(2)(3)(4)(5)(6)(7)$ and $(17)(26)(35)(4)$. Now under the the identity element every coloring is fixed and there are k ways of coloring each of the 7 vertices so there are k^7 colorings fixed by e. Now a coloring which is fixed by $(17)(26)(35)(4)$ must have vertex 1 and vertex 7 colored the same, 2 and 6 must be colored the same, 3 and 5 must be colored the same and 4 can be colored independently. Since there are 4 different groups to color, in total $Fix((17)(26)(35)(4)) = k^4$ so Burnside's Lemma says that there are

$$\frac{1}{2}(k^7 + k^4)$$

different unique colorings of this graph. It is not clearly obvious that this result is even an integer for all values of k, but it can be checked both for k even and for k odd that the result is always an integer. If $k = 1$ we see for sure that the formula works because there is then exactly $1 = \frac{1}{2}(1 + 1)$ ways of coloring the graph with one color.

Great, now that we have three examples of how this formula works, I want to justify why it is true. Fortunately it is a short calculation from the orbit-stabilizer theorem.

I need to introduce one bit of shorthand notation. Define

$$Fix(g) = \#\{x : g \bullet x = g\}$$

so then Burnside's Lemma can then be restated as

The number of orbits when G acts on $X = \frac{1}{|G|} \sum_{g \in G} Fix(g)$.

In order to make the first part of my calculation clear I am going to make a table. Along the top of the table I label the columns by the x_i which are in the set $X = \{x_1, x_2, x_3, \ldots, x_{|X|}\}$. Along the left side of the table I label the rows by g_i which are the elements of $G = \{g_1, g_2, \ldots, g_{|G|}\}$ and in the body of the table I put a mark $\times$ in row g and column x in my table if x is fixed by g (that is, if $g \bullet x = x$).

So our table will typically look like the following where I am placing the $\times$ symbols in the table in a way to indicate that for the average group element, some elements are fixed and some are not. For the identity group element all elements are fixed (this is by the definition of group action).

$G\backslash X$	x_1	x_2	x_3	$\cdots$	$x_{\|X\|}$
$e = g_1$	$\times$	$\times$	$\times$	$\cdots$	$\times$
g_2		$\times$		$\cdots$	
g_3		$\times$	$\times$	$\cdots$	$\times$
g_4	$\times$		$\times$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$g_{\|G\|}$		$\times$		$\cdots$	$\times$

Now in the right hand column of the table I will count how many $\times$ symbols there are in each row. I have already given this quantity a name. The number of $\times$ symbols in the row indexed by g_i is $Fix(g_i)$, the number of elements of my set X which are fixed by g_i.

$G\backslash X$	x_1	x_2	x_3	$\cdots$	$x_{\|X\|}$	
$e = g_1$	$\times$	$\times$	$\times$	$\cdots$	$\times$	$Fix(g_1)$
g_2		$\times$		$\cdots$		$Fix(g_2)$
g_3		$\times$	$\times$	$\cdots$	$\times$	$Fix(g_3)$
g_4	$\times$		$\times$	$\cdots$		$Fix(g_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	
$g_{\|G\|}$		$\times$		$\cdots$	$\times$	$Fix(g_{\|G\|})$

Now below each column I will tally how many symbols $\times$ which appear in each column. This quantity has also been given a name. The number of $\times$ symbols which appear in the column indexed by x_i is the number of group elements which fix x_i or it is the number of elements in the stabilizer of x_i, $|Stab(x_i)|$

$G\backslash X$	x_1	x_2	x_3	$\cdots$	$x_{\|X\|}$	
$e = g_1$	$\times$	$\times$	$\times$	$\cdots$	$\times$	$Fix(g_1)$
g_2		$\times$		$\cdots$		$Fix(g_2)$
g_3		$\times$	$\times$	$\cdots$	$\times$	$Fix(g_3)$
g_4	$\times$		$\times$	$\cdots$		$Fix(g_4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$g_{\|G\|}$		$\times$		$\cdots$	$\times$	$Fix(g_{\|G\|})$
	$\|Stab(x_1)\|$	$\|Stab(x_2)\|$	$\|Stab(x_3)\|$	$\cdots$	$\|Stab(x_{\|X\|})\|$	

So now if I sum the last row of this table it is equal to the total number of $\times$ symbols in the table and if I sum the last column it is also equal to the total number of $\times$ symbols in the table, hence we have that:

$$\sum_{x \in X} |Stab(x)| = \sum_{g \in G} Fix(g) \tag{2}$$

The right hand side of this equality is the right hand side of Burnside's Lemma multiplied by $|G|$. We also know from the orbit-stabilizer theorem that $|Stab(x)| = \frac{|G|}{|O_x|}$. Say that the set X breaks down into various orbits under the action of G and we number the orbits by a single representative:

$$X = O_{x_1} \uplus O_{x_2} \uplus O_{x_3} \uplus \cdots \uplus O_{x_{total\ \#\ orbits}}$$

Now then the left hand side of equation (2) is equal to

$$\begin{aligned}
\sum_{x \in X} |Stab(x)| &= \sum_{i=1}^{total\ \#\ orbits} \sum_{x \in O_{x_i}} |Stab(x)| \\
&= \sum_{i=1}^{total\ \#\ orbits} \sum_{x \in O_{x_i}} \frac{|G|}{|O_x|} \\
&= |G| \sum_{i=1}^{total\ \#\ orbits} \sum_{x \in O_{x_i}} \frac{1}{|O_{x_i}|} \\
&= |G| \sum_{i=1}^{total\ \#\ orbits} \frac{|O_{x_i}|}{|O_{x_i}|} \\
&= |G| \sum_{i=1}^{total\ \#\ orbits} 1 \\
&= |G| \cdot total\ \#\ orbits
\end{aligned}$$

Therefore we have show that $|G| \cdot total\ \#\ orbits = \sum_{g \in G} Fix(g)$, so

$$total\ \#\ orbits = \frac{1}{|G|} \sum_{g \in G} Fix(g)$$

Before I finished for the day I tried to squeeze in one more explanation. I wanted in fact to explain the example with the coloring with squares from the example above, and in particular I wanted to provide you with a formula for the generating function in equation (1).

Burnside's Lemma is quite robust because it just talks about a set X and it can be any set of colorings with a group action on them. The thing about group actions when they act on colorings is that the number of colors is independent of the element of the group acting on it so Burnside's Lemma says:

$$total\ \#\ orbits\ of\ colorings\ with\ a_i\ of\ i^{th}\ color\ appearing = \frac{1}{|G|} \sum_{g \in G} Fix_{with\ a_i\ color\ i}(g)$$

where $Fix_{with\ a_i\ color\ i}(g)$ represents the number of colorings with a_1 of color 1, a_2 of color 2, a_3 of color 3, etc. and the phrase *total # orbits of colorings with* a_i *of* i^{th} *color appearing* represents the subset of all of the distinct colorings with a_1 of color 1, a_2 of color 2, a_3 of color 3, etc. Because the group action does not affect the number of each color that appears, Burnside's Lemma applies.

Now sum over all weights $a = (a_1, a_2, a_3, \ldots)$ and multiply by $z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots$.

$$\sum_a (total\ \#\ orbits\ of\ colorings\ with\ a_i\ of\ i^{th}\ color\ appearing)\ z_1^{a_1}\ z_2^{a_2} z_3^{a_3} \cdots$$

$$= \sum_a \left(\frac{1}{|G|} \sum_{g \in G} Fix_{with\ a_i\ color\ i}(g) \right) z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_a (Fix_{with\ a_i\ color\ i}(g)\ z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots)$$

This is Polya's Theorem.

The left hand side of this equation is called the pattern inventory of the set. It is the generating function for the number of colorings where the coefficient of $z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots$ is the number of colorings with a_1 of color 1, a_2 of color 2, a_3 of color 3, etc.

The piece of the generating function $\sum_a (Fix_{with\ a_i\ color\ i}(g)\ z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots)$ on the right hand side is called the cycle index polynomial. If you look at it in one light Polya's Theorem *is* Burnside's Lemma with just a generating function replacing a number.

What is ingenious about this formula is that once we have the cycle structure of the group element g, the cycle index polynomial is usually very easy to compute because we can apply the multiplication principle of generating functions on the cycles. That is the generating function for the cycle index polynomial of g which is a product of cycles c_1, c_2, c_3, etc. is equal to the product of the cycle index polynomial for c_1 times the the cycle index polynomial for c_2 times the cycle index polynomial for c_3 times etc.

For instance, consider again the group of the cube and colorings with two colors B and W. Instead of the variables z_1 and z_2 I am going to use B and W in my cycle index polynomial to make it clearer which is the first color and the second color.

With the identity element $e = (A)(B)(C)(D)(E)(F)$, we have that

$$\sum_{i=0}^{6} (\#\text{colorings fixed by } e \text{ with } i \text{ W's } 6-i \text{ B's})\ W^i B^{6-i} = (B+W)^6.$$

There are two ways of deducing this. The first is to say that the number of colorings with i white faces and $6-i$ black faces is equal to $\binom{6}{i}$ and $\sum_{i=0}^{6} \binom{6}{i} W^i B^{6-i} = (B+W)^6$. The other way to deduce it is to say that it is equal to the product of the generating function for the colorings of the face A times the generating function for the colorings of the face B times $\cdots$ the generating function for the number of colorings of the face $F = (B+W)^6$.

Consider the element $(A)(D)(BCEF)$. The generating function for the colorings which are fixed by this element is the product of the generating function for the colorings of the face A times the generating function for the colorings of the face D times the generating function for the colorings of B, C, E and F. These last 4 need to be done together because they all need to be the same to color. Therefore

$$\sum_a (\#\text{colorings fixed by } (A)(D)(BCEF) \text{ with } a_1 \text{ W's } a_2 \text{ B's})\, W^{a_1} B^{a_2} = (B+W)^2(B^4+W^4).$$

Unless there are extra conditions placed on the colorings, it is easy to write down the generating function for the colorings of a group element with cycles of length $r_1, r_2, \ldots, r_\ell$ because each cycle will have the same color so the generating function for a cycle of size r_1 is always $B^{r_1} + W^{r_1}$ and the generating function for the colorings which are fixed by g is

$$(B^{r_1} + W^{r_1})(B^{r_2} + W^{r_2}) \cdots (B^{r_\ell} + W^{r_\ell}) .$$

Therefore the rest of group elements have cycle index polynomials

- $(B + W)^6$ for one identity element $(A)(B)(C)(D)(E)(F)$
- $(B+W)^2(B^4+W^4)$ six rotations about two fixed faces by $\pm 90°$ (e.g. $(A)(D)(BCEF)$)
- $(B+W)^2(B^2+W^2)^2$ three rotations about two fixed faces by $180°$ (e.g. $(A)(D)(BE)(CF)$)
- $(B^2 + W^2)^3$ six flips about an edge (e.g. $(AB)(DE)(CF)$)
- $(B^3 + W^3)^2$ eight rotations about a vertex by $\pm 120°$ (e.g. $(ABC)(DEF)$)

Therefore, the generating function for the colorings with black and white faces is given by the expression

$$\frac{1}{24}((B+W)^6+6(B+W)^2(B^4+W^4)+3(B+W)^2(B^2+W^2)^2+6(B^2+W^2)^3+8(B^3+W^3)^2)$$

I asked Sage to expand this result for me and I find that it gives exactly the result in equation (1).

```
sage: B,W = var('B','W')
sage: expand(1/24*((B+W)^6 + 6*(B+W)^2*(B^4+W^4) + 3*(B+W)^2*(B^2+W^2)^2 + \
6*(B^2+W^2)^3 + 8*(B^3+W^3)^2))
B^6 + B^5*W + 2*B^4*W^2 + 2*B^3*W^3 + 2*B^2*W^4 + B*W^5 + W^6
```

I can more or less count the number of colorings of the cube with two colors by hand, but increasing the number of colors or the size of the object does not significantly increase the complexity of using this formula but it does make counting these colorings by hand significantly more complicated. Consider colorings of the cube with three colors (just as an example).

```
sage: R,G,B = var('R,G,B')
sage: expand(1/24*((R+G+B)^6 + 6*(R+G+B)^2*(R^4+G^4+B^4) \
+ 3*(R+G+B)^2*(R^2+G^2+B^2)^2 + 6*(R^2+G^2+B^2)^3 + 8*(R^3+G^3+B^3)^2))
B^6 + B^5*G + B^5*R + 2*B^4*G^2 + 2*B^4*G*R + 2*B^4*R^2 + 2*B^3*G^3
+ 3*B^3*G^2*R + 3*B^3*G*R^2 + 2*B^3*R^3 + 2*B^2*G^4 + 3*B^2*G^3*R + 6*B^2*G^2*R^2
+ 3*B^2*G*R^3 + 2*B^2*R^4 + B*G^5 + 2*B*G^4*R + 3*B*G^3*R^2 + 3*B*G^2*R^3
+ 2*B*G*R^4 + B*R^5 + G^6 + G^5*R + 2*G^4*R^2 + 2*G^3*R^3 + 2*G^2*R^4 + G*R^5 + R^6
```

NOTES ON NOV 20, 2012

MIKE ZABROCKI

This class I started by asking the question:

> How many ways are there of placing n colored beads made up of k colors around a necklace if you can slide these beads around the necklace, but not turn it over?

You can imagine a necklace with large beads that hang from the cord and that these beads can slide from one side of the necklace to the other. Two colored necklaces are equal if you can rotate the beads around the necklace so that the necklaces are the same.

I did this on an example of 8 beads. To apply Burnside's Lemma we need to recognize that there is a group of motions of rotations of the beads acting on the necklace. In the example of $n = 8$, the 8 group elements can be represented as the permutations in the table below. Let R_r be a rotation of r beads from the left side to the right side.

$g \in G$	cycle notation
$R_0 = R_8$	$(1)(2)(3)(4)(5)(6)(7)(8)$
R_1	(18765432)
R_2	$(1753)(2864)$
R_3	(16385274)
R_4	$(15)(26)(37)(48)$
R_5	(147258361)
R_6	$(1357)(2468)$
R_7	(12345678)

What I note is that there are four elements with one cycle of length 8 $\{R_1, R_3, R_5, R_7\}$, two elements with two cycles of length 4 $\{R_2, R_6\}$, one element with four cycles of length 2 $\{R_4\}$, and one element with eight cycles of length 1 $\{R_0 = R_8\}$. We conclude that the formula for making necklaces with 8 beads and k colors is equal to

$$\frac{1}{8}(k^8 + k^4 + 2k^2 + 4k)$$

because on each of the cycles we have k choices to use for the colors.

At this point I wrote down the formula that I know is true in general and I said that we should observe that it works in this case.

The number of ways of making an n bead necklace with k colored beads when you are not allowed to turn over the necklace but you can rotate the beads is equal to

$$\frac{1}{n}\sum_{d|n} \phi(d) k^{n/d}$$

where $\phi(d)$ is equal to the number of integers which is relatively prime to d (an integer n is relatively prime to d if $gcd(n, d) = 1$). In order to see why this might be true I made the following table.

d	integers between 1 and d which are relatively prime to d	motions which have n/d cycles of length d
8	$\{1, 3, 5, 7\}$	$\{R_1, R_3, R_5, R_7\}$
4	$\{1, 3\}$	$\{R_2, R_6\}$
2	$\{1\}$	$\{R_4\}$
1	$\{1\}$	$\{R_8\}$

What we want to do is show that this holds in general. If I define $\Phi(d)$ to be the integers between 1 and d which are relatively prime to d and $\Psi(d)$ be the indices i such that R_i is made up n/d cycles of length d. I claim that there is a bijection between $\Phi(d)$ and $\Psi(d)$. The bijection is simple $V_d(x) = \frac{n}{d}x$ is a map from the elements of $\Phi(d)$ to the elements of $\Psi(d)$ (notice in the table above to go from the set $\Phi(8) = \{1, 3, 5, 7\}$ to the set $\Psi(8) = \{i : R_i \in \{R_1, R_3, R_5, R_7\}\}$ we multiply each element by 1; to go from $\Phi(4) = \{1, 3\}$ to $\Psi(4) = \{i : R_i \in \{R_2, R_6\}\}$ we multiply each element by 2; to go from $\Phi(2) = \{1\}$ to $\Psi(2) = \{i : R_i \in \{R_4\}\}$ we multiply the element by 4; to go from $\Phi(1) = \{1\}$ to $\Psi(1) = \{i : R_i \in \{R_8\}\}$ we multiply the element by 8.

At this point there was some detail that I was missing (a basic fact about integers) and I totally blanked, so we moved on. I promised to come back to it and explain precisely why this was a bijection.

The next thing I wanted to do was give a formula for the number of elements with a given cycle structure. The reason that we might need to do this is because Burnside's Lemma and Polya's theorem requires that we sum over the group elements and the quantities $Fix(g)$ or the generating function $\sum_a (Fix_{with\ a_i\ color\ i}(g)\ z_1^{a_1} z_2^{a_2} z_3^{a_3} \cdots)$. There is a formula for the number of permutations with a given cycle structure and it uses the orbit-stabilizer theorem.

First we have to define an action on permutations. Assume that π is a permutation that when written in cycle notation has the following form:

$$\pi = (i_1 i_2 \cdots i_r)(j_1 j_2 \cdots j_s) \cdots (\ell_1 \ell_2 \cdots \ell_d)$$

then we computed $g \circ \pi \circ g^{-1}$. When we compute $g \circ \pi \circ g^{-1}$ on an element x, first we apply g^{-1} to x and get $x' = g^{-1}(x)$, then we apply π to x' to get $x'' = \pi(x')$ and then we apply g to x'' to get $g(x'')$ It doesn't matter which element we start our computation with, so for no other reason than it will work out nice, start with $g(i_1)$. In this case

$$\begin{array}{ccc} g(i_1) & \xrightarrow{g^{-1}} & i_1 \\ i_1 & \xrightarrow{\pi} & i_2 \\ i_2 & \xrightarrow{g} & g(i_2) \end{array}$$

so we have that $g(i_1)$ is sent to $g(i_2)$.

$$g(i_2) \xrightarrow{g^{-1}} i_2$$

$$i_2 \xrightarrow{\pi} i_3$$
$$i_3 \xrightarrow{g} g(i_3)$$

then we see that $g(i_2)$ is sent to $g(i_3)$. And $g(i_r)$ will be sent to $g(i_1)$ and in general $g(i_a)$ will be sent to $g(i_{a+1})$. This says that the first cycle of $g \circ \pi \circ g^{-1} = (g(i_1)g(i_2)\cdots g(i_r))\cdots$. Then with a similar argument $g \circ \pi \circ g^{-1}(g(j_a)) = g(j_{a+1})$ and the rest of the permutation $g \circ \pi \circ g^{-1}$ can be written in cycle notation as

$$g \circ \pi \circ g^{-1} = (g(i_1)\ g(i_2)\cdots g(i_r))(g(j_1)\ g(j_2)\cdots g(j_s))\cdots(g(\ell_1)\ g(\ell_2)\cdots g(\ell_d))$$

This was a tricky point, so I suggested that we figure this out on an example. We have done some compositions of permutations, but not a lot. Take as an experiment a permuation

$$\pi = (152)(3748)(6)$$

and $g = (14382)(56)(7)$ and compute $g \circ \pi \circ g^{-1}$ in two different ways. In one way, compute $g \circ \pi \circ g^{-1}(1)$, $g \circ \pi \circ g^{-1}(2)$, etc. and figure out the cycle structure. In another calculation, compute

$$(g(1)\ g(5)\ g(2))(g(3)\ g(7)\ g(4)\ g(8))(g(6)) = (461)(8732)(5)$$

and verify that you have the same permutation.

Now we can verify that g acting on π by $g \circ \pi \circ g^{-1}$ a group action. Remember that a group action has to satisfy two axioms, $e \bullet \pi = \pi$ and $g \bullet (h \bullet \pi) = (g \circ h) \bullet \pi$. In this case

$$e \bullet \pi = e \circ \pi \circ e = \pi$$

and

$$g \bullet (h \bullet \pi) = g \bullet (h \circ \pi \circ h^{-1}) = g \circ (h \circ \pi \circ h^{-1}) \circ g^{-1} = (gh) \circ \pi \circ (h^{-1} \circ g^{-1})$$

It is not too hard to verify that $(h^{-1} \circ g^{-1}) = (g \circ h)^{-1}$ since $(g \circ h) \circ (g \circ h)^{-1} = e$ and $(g \circ h) \circ (h^{-1} \circ g^{-1}) = e$. Hence $g \bullet (h \bullet \pi) = (g \circ h) \bullet \pi$. This group action is called 'conjugation.'

What this says is that the structure of the cycles of the permutations is preserved by the group action of conjugation. If you look closely it also says that for any two permutations with the same cycle structure, there is a permuation g which takes one to the other under the action of conjugation. For instance $(152)(3748)(6)$ and $(461)(8732)(5)$ have the same cycle structure and there is a permutation which takes one to the other, but any other permutatation with the same number of cycles of each length (say $(123)(4567)(8)$) is also in the orbit of these two permutations. In fact, the original question that I asked can now be rephrased as a question about the size of the orbit. Let me state it more precisely:

How many permutations have a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc.?

Alternatively, how many permutations are in the orbit of a permutation with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc. under the action of conjugation?

The answer is to use the orbit stabilizer theorem which says that now that we have the action of the group of permutations on π, if we divide $n!$ (the number of all permutations by the number of permutations g for which $g \bullet \pi = \pi$, then we will have the number of elements in the orbit of π.

Lets do this on an example of a permutation of 4 with two cycles of length 2.

The following permutations are all the same (12)(34), (21)(34), (12)(43), (21)(43), (34)(12), (43)(12), (34)(21), (43)(21) and each one of these permutations has a different g which sends (12)(34) to each of them, namely (1)(2)(3)(4), (12)(3)(4), (1)(2)(34), (12)(34), (13)(24), (1423), (1324), (14)(23). These permutations are the stabilizer of (12)(34) under the action of conjugation. This means that the orbit of (12)(34) is $4!/8 = 3$ and we know that there are three permutations with 2 cycles of length 2, namely (12)(34), (13)(24) and (14)(23).

What would happen if we had a_2 of cycles of length 2? say $(12)(34)(56)\cdots(2a_2-1, 2a_2)$? Well there are $a_2!$ ways of permuting the cycles and $(i, i+1)$ can be sent to $(j, j+1)$ or $(j+1, j)$ for each of the a_2 cycles so there are $2^{a_2}a_2!$ permutations g in the stabilizer of this permutation.

I then started rushing because I realized that I was more or less out of time. If there are a_3 cycles of length 3 then each of the a_3 cycles can be rearranged and $(i, i+1, i+2)$ can be sent either to $(j, j+1, j+2)$, $(j+1, j+2, j)$ or $(j+2, j, j+1)$ and all three of these cycles are exactly the same. Hence there are $a_3!3^{a_3}$ permutations in the orbit of $(123)(456)\cdots(3a_3-2, 3a_3-1, 3a_3)$.

In general I said that if there are a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc. then there are

$$a_1!1^{a_1}a_2!2^{a_2}a_3!3^{a_3}\cdots = \prod_{i\geq 1} a_i!i^{a_i}$$

elements in the stabilizer by conjugation and

$$n!/(a_1!1^{a_1}a_2!2^{a_2}a_3!3^{a_3}\cdots)$$

permutations with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc.

NOTES ON NOV 22, 2012

MIKE ZABROCKI

I had a tight agenda for this class because there were a couple of questions on the homework that people asked me about.

First, the questions about how to prove the generating functions $B(x,u) = \sum_{n\geq 0}\sum_{k=1}^{n} S(n,k)u^k\frac{x^n}{n!}$ is equal to $e^{u(e^x-1)}$. I had talked with a few people after class and they wanted to know how to do this problem because it wasn't exactly like what I had done in class for $B(x) = \sum_{n\geq 0} B_n\frac{x^n}{n!}$ to show that $B(x) = e^{e^x-1}$. I said, well if you worked on this problem you should have all found that

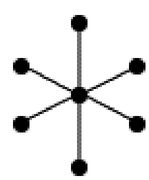
$$\frac{\partial}{\partial x}B(x,u) = uB(x,u) + u\frac{\partial}{\partial u}B(x,u) \ .$$

Then, once you do this, you have more or less shown that $B(x,u) = e^{u(e^x-1)}$ after you also show that

$$\frac{\partial}{\partial x}(e^{u(e^x-1)}) = u(e^{u(e^x-1)}) + u\frac{\partial}{\partial u}(e^{u(e^x-1)}) \ .$$

Why? Think about it and I will explain next time why this (plus one or two minor details) proves that $B(x,u) = e^{u(e^x-1)}$.

The next thing I discussed was the number of ways of coloring the spoked graph:



A motion of this graph which preserves the structure can permute any of the six outer vertices but the center vertex must be fixed. Once we know how many cycles of each type there are, we color each of the cycles with k colors. The number of colorings which are fixed by a permutation with d cycles is equal to k^d. But recall that at the beginning of the class we computed a formula for the number of permutations of n with d cycles and this was the unsigned Stirling numbers of the first kind.

In particular we had a table (see the notes from Sept 13-18 for a table of the signed Stirling numbers of the first kind but only up to $n = 4$). We also had a formula from the first homework assignment. We also have the last problem from homework #4 and the formula $\sum_{n\geq 0}\sum_{k=1}^{n} s'(n,k)u^k\frac{x^n}{n!} = e^{-u\ log(1-x)}$. I used the computer then to compute the unsigned Stirling numbers assuming that this formula is correct.

```
sage: (u,x) = var('u,x')
sage: taylor(exp(-u*log(1-x)),x,0,10)
1/3628800*(u^10 + 45*u^9 + 870*u^8 + 9450*u^7 + 63273*u^6 + 269325*u^5 +
 723680*u^4 + 1172700*u^3 + 1026576*u^2 + 362880*u)*x^10 + 1/362880*(u^9
 + 36*u^8 + 546*u^7 + 4536*u^6 + 22449*u^5 + 67284*u^4 + 118124*u^3 +
 109584*u^2 + 40320*u)*x^9 + 1/40320*(u^8 + 28*u^7 + 322*u^6 + 1960*u^5
 + 6769*u^4 + 13132*u^3 + 13068*u^2 + 5040*u)*x^8 + 1/5040*(u^7 + 21*u^6
 + 175*u^5 + 735*u^4 + 1624*u^3 + 1764*u^2 + 720*u)*x^7 + 1/720*(u^6 +
 15*u^5 + 85*u^4 + 225*u^3 + 274*u^2 + 120*u)*x^6 + 1/120*(u^5 + 10*u^4
 + 35*u^3 + 50*u^2 + 24*u)*x^5 + 1/24*(u^4 + 6*u^3 + 11*u^2 + 6*u)*x^4
 + 1/6*(u^3 + 3*u^2 + 2*u)*x^3 + 1/2*(u^2 + u)*x^2 + u*x + 1
```

This calculation says that (for instance) there are 15 permutations of 6 with 5 cycles. Because these permutations have one cycle of length 2 and 4 cycles of length 1 we know that there are $\binom{6}{2}$ possible permutations because this is the number of ways of choosing two elements to make a 2 cycle.

Once we know that a permutation has d cycles for the 6 outer vertices and one cycle of length 1 for the center vertex, then there are k^{d+1} ways of coloring the graph so that it is fixed by that permutation.

If we were to write down the formula for the colorings of the graph above it would be

$$\frac{1}{6!}\sum_{d=1}^{6} s'(6,d)k^{d+1} = \frac{1}{720}(k^7 + 15k^6 + 85k^5 + 225k^4 + 274k^3 + 120k^2)$$

This argument is quite general and it also says that in fact that the number of colorings of the spoke graph where there are n spokes coming off of a center vertex is given by

$$\frac{1}{n!}\sum_{d=1}^{n} s'(n,d)k^{d+1} \ .$$

I should remind you that in the first homework assignment we showed that $(k)^{(n)} = k(k+1)(k+2)\cdots(k+(n-1)) = \sum_{d=1}^{n} s'(n,d)k^d$, hence the number of colorings of the spoke graph with n spokes is equal to

$$\frac{1}{n!}\sum_{d=1}^{n} s'(n,d)k^{d+1} = \frac{k(k)^{(n)}}{n!} \ .$$

We can't really see this factorization on the sage example above because we have to tell the computer to factor each of the coefficients of the series if that is what we want.

```
sage: f = taylor(exp(-u*log(1-x)),x,0,10)
sage: sum(factor(f.coefficient(x,i))*x^i for i in range(11))
1/3628800*(u + 1)*(u + 2)*(u + 3)*(u + 4)*(u + 5)*(u + 6)*(u + 7)*(u
 + 8)*(u + 9)*u*x^10 + 1/362880*(u + 1)*(u + 2)*(u + 3)*(u + 4)*(u +
 5)*(u + 6)*(u + 7)*(u + 8)*u*x^9 + 1/40320*(u + 1)*(u + 2)*(u +
 3)*(u + 4)*(u + 5)*(u + 6)*(u + 7)*u*x^8 + 1/5040*(u + 1)*(u + 2)*(u
```

```
 + 3)*(u + 4)*(u + 5)*(u + 6)*u*x^7 + 1/720*(u + 1)*(u + 2)*(u + 3)*(u
 + 4)*(u + 5)*u*x^6 + 1/120*(u + 1)*(u + 2)*(u + 3)*(u + 4)*u*x^5 +
1/24*(u + 1)*(u + 2)*(u + 3)*u*x^4 + 1/6*(u + 1)*(u + 2)*u*x^3 +
1/2*(u + 1)*u*x^2 + u*x + 1
```

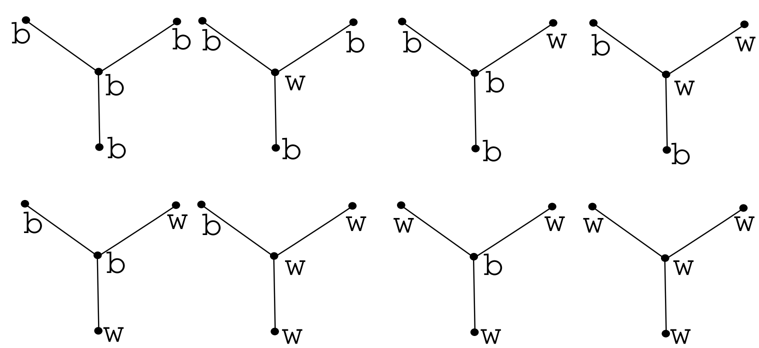
Moreover we can also not only give the formula for a single entry in this sequence, we can use this formula to give the generating function for all possible colorings of all the spoke graphs at the same time.

$$\begin{aligned}
&\text{g.f. for colorings of spoke graph with } n \text{ spokes} \\
&= \sum_{n\geq 0} (\text{number of colorings of } n \text{ spoke graph}) x^n \\
&= \sum_{n\geq 0} \frac{1}{n!} \sum_{d=1}^{n} s'(n,k) k^{d+1} x^n \\
&= k \sum_{n\geq 0} \sum_{d=1}^{n} s'(n,k) k^{d+1} \frac{x^n}{n!}
\end{aligned}$$

You should notice however that this is precisely the generating function $ke^{-klog(1-x)}$ given in the last problem of the homework. So for instance lets say that we set $k = 2$ and look at the computer expansion of this series.

```
sage: taylor(2*exp(-2*log(1-x)),x,0,10)
22*x^10 + 20*x^9 + 18*x^8 + 16*x^7 + 14*x^6 + 12*x^5 + 10*x^4 + 8*x^3
 + 6*x^2 + 4*x + 2
```

This shows that for instance the graph with three spokes coming off of a center vertex can be colored with $k = 2$ colors in 8 different ways so that the colorings are distinctly different. For example:



are all possible colorings of the graph with three spokes coming off of a center vertex. What is very cool is that this homework problem relates the problems in HW #1 part 1 question 2,3 and HW #4 part 1 question 1 and HW # 4 part 2 question 3.

So the next thing that I wanted to discuss was the number of permutations with a given cycle structure. I rushed through the explanation at the end of class last time and I wanted to give a few more details about the formula that I stated very quickly.

Say that we want to know how many permutations there with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc. This means that

$$n = a_1 + 2a_2 + 3a_3 + \cdots$$

There are only a finite number of solutions to this equation for any fixed n. In particular there is one for every partition of n. This is because the lengths of the cycles of the permutation determine a partition (e.g. (123)(456)(78)(9) and (1)(234)(56)(789) both determine the partition $(3, 3, 2, 1)$ by the lengths of their cycles since the order of the cycles is not important).

So lets say that we wanted to count the number of permutations with a_i cycles of length i for $i \geq 1$. In this case we want to find the orbit of the permutation:

$$\pi = (1)(2)\cdots(b_1)(b_1+1, b_1+2)\cdots(b_2-1, b_2)(b_2+1, b_2+2, b_2+3)\cdots(b_3-2, b_3-1, b_3)\cdots$$

where $b_1 = a_1$, $b_2 = a_1 + 2a_2$, $b_3 = a_1 + 2a_2 + 3a_3$, and in general $b_r = \sum_{i=1}^{r} a_i$. This is a permutation with a_1 cycles of length 1, a_2 cycles of length 2, a_3 cycles of length 3, etc.

Now I want to make a procedure which determines another representation of this permutation which is equivalent. That is I want to determine an element g such that $g \circ \pi \circ g^{-1} = \pi$.

- The a_1 cycles of length 1 may be sent to a permutation of the a_1 cycles.
- The a_2 cycles of length 2 are of the form $(i, i+1)$ and they may be sent to a permutation of the cycles and each one may be sent to either a cycle of the form $(j, j+1)$ or $(j+1, j)$.
- The a_3 cycles of length 3 are of the form $(i, i+1, i+2)$ and they may be sent to a permutation of the cycles and each one may be sent to one of the form $(j, j+1, j+2)$, $(j+1, j+2, j)$ or $(j+2, j, j+1)$
- In general, the a_r cycles of length r are of the form $(i, i+1, \ldots, i+r-1)$ they may be sent to a permutation of the a_r cycles and each one may be sent to one of the r different cyclic shifts of the cycles $(i+d, i+d+1, \ldots, i+r-1, i, i+1, \ldots, i+d-1)$.

This describes a procedure for determining one possible permutation g which is in the stabilizer of π. The number of outcomes of the r^{th} step of this procedure is $a_r! r^{a_r}$ since there are $a_r!$ ways of permuting the cycles and r choices for each one of the cycles as to how it is shifted. Hence, by the multiplication principle the number of elements in the stabilizer of π is equal to

$$a_1! 1^{a_1} a_2! 2^{a_2} a_3! 3^{a_3} \cdots = \prod_{r \geq 0} a_r! r^{a_r} \ . \tag{1}$$

Now since the number of elements in the stabilizer of π is equal to equation (1), the number of elements in orbit of this group action is equal to $n!$ divided by the number of elements in the stabilizer,

$$\frac{n!}{\prod_{r \geq 0} a_r! r^{a_r}} \ .$$

The number of elements in the orbit of this element is equal to the number of permutations with a_r cycles of length r for $r \geq 1$.

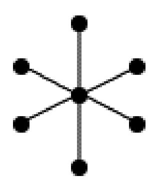
Lets do this for permutations of 6 and verify that it agrees with what we computed for the unsigned Stirling numbers that we computed from the generating function earlier.

permutation π	$\#\{g \in S_6 : g \circ \pi \circ g^{-1} = \pi\}$	$\#\{g \circ \pi \circ g^{-1} : g \in S_6\}$
(1)(2)(3)(4)(5)(6)	$6! = 720$	$720/720 = 1$
(1)(2)(3)(4)(56)	$4!2 = 48$	$720/48 = 15$
(1)(2)(34)(56)	$2!2!2^2 = 16$	$720/16 = 45$
(1)(2)(3)(456)	$3!3 = 18$	$720/18 = 40$
(12)(34)(56)	$3!2^3 = 48$	$720/48 = 15$
(1)(23)(456)	$2 \cdot 3 = 6$	$720/6 = 120$
(1)(2)(3456)	$2!4 = 8$	$720/8 = 90$
(123)(456)	$2!3^2 = 18$	$720/18 = 40$
(12)(3456)	$2 \cdot 4 = 8$	$720/8 = 90$
(1)(23456)	5	$720/5 = 144$
(123456)	6	$720/6 = 120$

We can use this table to compute that $s'(6,5) = 15$, $s'(6,4) = 45 + 40 = 85$, $s'(6,3) = 15 + 120 + 90 = 225$, $s'(6,2) = 40 + 90 + 144 = 274$, $s'(6,1) = 120$.

You might ask (and someone did) if we can use the Stirling numbers to compute the number of permutations with a given number of cycles, then why do we need to know how to find the number of permutations with a given cycle type? If you need to apply Polya's theorem rather than Burnside's Lemma (a formula which contains more detailed information), then you need to know precisely the number of cycles of each type for each of the permutations and not just how many cycles there are. In particular I can ask more pointed questions like the following:

How many colorings of the graph



are there using k colors such that each color is used at most twice?

NOTES ON NOV 27, 2012

MIKE ZABROCKI

I answered questions about the homework problems. One of them was the second problem about generating functions. Someone simply asked 'how do you do it?' It is hard to give a hint on this one with out shoving you in the right direction. The problem was to prove a formula (which was part of the problem but I don't remember what it is right now) for the $S(n,k)$. But in problem number (1) you were asked to show that the generating function for $S(n,k)$ is equal to $e^{u(e^x-1)}$. Armed with this piece of information you know that the coefficient of $u^k\frac{x^n}{n!}$ in $e^{u(e^x-1)}$ is $S(n,k)$, but you also have that

$$e^{u(e^x-1)} = \sum_{d\geq 0} u^d \frac{(e^x-1)^d}{d!}$$

Now you should notice that you can use the binomial theorem to expand $(e^x-1)^d = \sum_{i=0}^d \binom{d}{i}(-1)^i e^{(d-i)x}$ and now take the coefficient of $u^k\frac{x^n}{n!}$ in the expression you get there. At this point there isn't too much left to do but remember that the coefficient of $\frac{x^n}{n!}$ in e^{cx} is equal to c^n.

Then I knew that I wanted to talk a little bit about the first and the third problems in that section. I said in the last class that 'all' you had to do was show that $B(x,u) := 1 + \sum_{n\geq 1}\sum_{k=1}^n S(n,k)u^k\frac{x^n}{n!}$ satisfied the differential equation

$$\frac{\partial}{\partial x}B(x,u) = uB(x,u) + u\frac{\partial}{\partial u}B(x,u)$$

and you were done, but that is a little inaccurate. It is the major step of the proof, but there is an argument to be made to verify that you really are done.

The coefficients $S(n,k)$ are defined by the recurrence $S(n+1,k) = S(n,k-1)+kS(n,k)$ for $n\geq 0$ and $k\geq 1$ and the initial conditions that $S(0,0)=1$ and $S(n,0)=S(0,n)=0$ for $n>0$. What you need to do is show that the coefficients in the series for $e^{u(e^x-1)}$ also satisfies the same defining relations.

There are three steps that you need to complete in order to show this. First, let $V(x,u)$ be a function with a taylor series $V(x,u) = \sum_{n,k\geq 0} a_{n,k}u^k\frac{x^n}{n!}$ and show that $V(x,u)$ satisfies

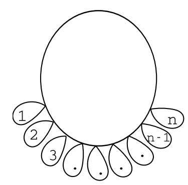
$$\frac{\partial}{\partial x}V(x,u) = uV(x,u) + u\frac{\partial}{\partial u}V(x,u)$$

if and only if

$$a_{n+1,k} = a_{n,k-1} + ka_{n,k}$$

for $n \geq 0$ and $k \geq 1$. This is more or less exactly what you needed to do in order to show that $B(x,u)$ satisfies this equation, but you also need to go backwards. Second you need to show that $e^{u(e^x-1)}$ satisfies this differential equation. This is a relatively easy calculus calculation. Finally, you need to show that the coefficients satisfies the same base case. This amounts to showing $B(x,u)\Big|_{x^0} = e^{u(e^x-1)}\Big|_{x^0}$ and $B(x,u)\Big|_{u^0} = e^{u(e^x-1)}\Big|_{u^0}$. Since, both of these coefficients is equal to 1, you have shown that the coefficients satisfy the same base case and hence $a_{n,k} = S_{n,k}$ for all $n, k \geq 0$.

Next I talked about the formula for necklaces. I drew the picture of a necklace with beads hanging from a chain and I indicated that the motions of the necklace were R_r for $1 \leq r \leq n$ where this means take r beads from the right hand side and move them to the left hand side (note: for convenience I switched directions from the notation I used on November 20, but really this affects nothing significantly).



Notice what happens to bead number i under the action of R_r. Bead i is sent to $i+r$; then bead $i+r$ is sent to $i+2r$; bead $i+2r$ ends up where $i+3r$ was located; etc. This will make a cycle of length d when $i+dr$ ends up where bead i currently is. In order for this to happen dr must be a multiple of n (the total number of beads and this cycle will be exactly of length d if $dr = lcm(n,r)$.

There is a well known formula for $lcm(n,r)$ in terms of the greatest common divisor.

Lemma 1. *For positive integers a and b, $lcm(a,b) = \frac{ab}{gcd(a,b)}$.*

Take for example the $lcm(10,12) = 60$, this formula says it should be $10 \cdot 12 = 120$ divided by the $gcd(10,12) = 2$. I provided a quick proof of this fact just to convince you that it was true by looking at the prime factorizations of $a, b, gcd(a,b)$ and $lcm(a,b)$, but I won't bother to write it down here because it is based on the fundamental theorem of arithmetic and a few other properties of primes which I am assuming anyway. I might as well assume that this fact is true. There was another fact that I assumed was true that uses some properties of integers that I don't think that we will get into.

Lemma 2. *For positive integers c, d, e,*

$$gcd(d,e) = 1 \text{ if and only if } gcd(cd,ce) = c$$

Take again the example of $gcd(10,12) = 2$ and compare this to $gcd(5,6) = 1$.

Now I claim that I have enough information to write down the formula for the number of necklaces with n beads using k colors and this formula is written in terms of a quantity $\phi(d)$ = the number of integers e between 1 and d that are relatively prime to d.

$$\#\text{necklaces with } n \text{ beads colored with } k \text{ colors} = \frac{1}{n}\sum_{d|n}\phi(d)k^{n/d} \tag{1}$$

I am now thinking about it and I am not sure I mentioned why this is even useful. If you don't know a formula for $\phi(d)$, then we have given one formula that is hard to compute (Burnside's lemma) in terms of another (the formula in equation (1) in terms of $\phi(d)$). The thing is that there are formulas for $\phi(d)$. If d has a factorization into distinct primes $p_1^{a_1}p_2^{a_2}\cdots p_\ell^{a_\ell}$ then

$$\phi(d) = (p_1^{a_1} - p_1^{a_1-1})(p_2^{a_2} - p_2^{a_2-1})\cdots(p_\ell^{a_\ell} - p_1^{a_\ell-1}) .$$

For example $\phi(8) = 2^3 - 2^2 = 8 - 4 = 4$. But this is a side note.

There are n group elements which act on this necklace $R_1, R_2, R_3, \ldots, R_n = R_0 = e$. We have already deduced that R_r consists of cycles of length d if and only if $lcm(r,n) = rd$ and since $lcm(r,n) = rn/gcd(n,r)$ then it must be that the length of the cycle is $d = n/gcd(n,r)$ (verify that this actually happens on an example) and so $gcd(n,r) = n/d$.

But because of Lemma 2 above, we have that $gcd(n,r) = n/d$ if and only if $gcd(d, rd/n) = 1$. This means that for every $e = rd/n$ which is relatively prime to 1, there is an $r = \frac{n}{d}e$. This says that there is a bijection between the set $\Phi(d) = \{e : gcd(d,e) = 1\}$ and the set $\Psi(d) = \{r : gcd(n,r) = n/d\}$, and moreover the bijection from $\Phi(d)$ to $\Psi(d)$ is to multiply the elements of $\Phi(d)$ by n/d.

Therefore we know that there are $\phi(d) = |\Phi(d)|$ elements with n/d cycles of length d and so there are $k^{n/d}$ ways of coloring each of those n/d cycles. Burnside's Lemma then says that

$$\#\text{ necklaces} = \frac{1}{n}\sum_{r=1}^{n} Fix(R_r) = \frac{1}{n}\sum_{r=1}^{n} k^{gcd(n,r)} = \frac{1}{n}\sum_{d|n}\phi(d)k^{n/d} .$$

Recall that for our example of $n = 8$, we had the table of

$g \in G$	cycle notation
$R_0 = R_8$	(1)(2)(3)(4)(5)(6)(7)(8)
R_1	(18765432)
R_2	(1753)(2864)
R_3	(16385274)
R_4	(15)(26)(37)(48)
R_5	(147258361)
R_6	(1357)(2468)
R_7	(12345678)

And when we grouped them by the elements that consist of n/d cycles of length d. Then the following table agrees with this construction.

d = cycle length	integers between 1 and d that are relatively prime to d	motions which have n/d cycles of length d
8	$\{1,3,5,7\}$	$\{R_1, R_3, R_5, R_7\}$
4	$\{1,3\}$	$\{R_2, R_6\}$
2	$\{1\}$	$\{R_4\}$
1	$\{1\}$	$\{R_8\}$

For this example the ways of coloring a necklace with 8 beads and k colors is equal to

$$\frac{1}{8}(k^8 + k^4 + 2k^2 + 4k)$$

We can also apply Polya's theorem to get a refinement of this formula. Since the generating function for the ways of coloring a single cycle of length d is equal to $\sum_{i=1}^k x_i^d$, then by the multiplication principle of generating functions, the generating function for the number of ways of coloring n/d cycles of length d is equal to $\left(\sum_{i=1}^k x_i^d\right)^{n/d}$. Moreover, Polya's Theorem says that the generating function for the number of ways of coloring the necklaces with k colored beads will be

$$\frac{1}{n}\sum_{d|n}\phi(d)\left(\sum_{i=1}^k x_i^d\right)^{n/d} .$$

Lets try this in practice for $n = 8$, the generating function will be

$$\frac{1}{8}((R+B)^8 + (R^2+B^2)^4 + 2(R^4+B^4)^2 + 4(R^8+B^8))$$

Lets expand this with Sage (although I also did it by hand for a single coefficient):

```
sage: ( (R+B)^8 + (R^2+B^2)^4 + 2*(R^4 + B^4)^2 + 4*(R^8+B^8))/8
1/8*(B + R)^8 + 1/8*(B^2 + R^2)^4 + 1/4*(B^4 + R^4)^2 + 1/2*B^8 + 1/2*R^8
sage: expand(_)
B^8 + B^7*R + 4*B^6*R^2 + 7*B^5*R^3 + 10*B^4*R^4 + 7*B^3*R^5 + 4*B^2*R^6
 + B*R^7 + R^8
```

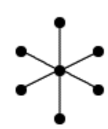
What this says is that there are 7 necklaces with 5 blue beads and 3 red beads, they are

$$BBBBBrrr,\ BBBBrBrr,\ BBBrBBrr,\ BBrBBBrr,$$
$$BrBBBBrr,\ BBBrBrBr,\ BBrBBrBr$$

Check very carefully and I THINK that all 7 of these are different and if they are, then every necklace is equivalent to one of these.

Next time I want you to work on the combinatorics problem that I posed last time:

How many colorings of the graph



are there using k colors such that each color is used at most twice?