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The definition of $\binom{n}{k}$ is the number of ways of picking k elements from an n element set. If $k < 0$ or $k > n$ this symbol is just the number 0. Recall that $n!$ is the number of ways of ordering n elements.

Proposition 1. *For $n \geq 0$ and $0 \leq k \leq n$,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \ .$$

Proof. Pick k elements to go first. Order the k elements. Order the last $n-k$ elements. By the multiplication principle, number of ways of doing this is $\binom{n}{k} k!(n-k)!$. Since this is also the number of ways of ordering all n elements it is equal to $n!$. Therefore, $\binom{n}{k} k!(n-k)! = n!$. □

But then I said, well we can also show this same result 'by induction.'

Lemma 2.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. Every way of picking k elements from an n element set either contains the largest element or it doesn't. Since a choice of the n elements that contains the largest consists of $k-1$ other elements from an $n-1$ element set, there are $\binom{n-1}{k-1}$ ways of doing this. There are also $\binom{n-1}{k}$ ways of picking k elements that do not contain the largest. By the addition principle, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. □

Proposition 3. *For $n \geq 0$ and $0 \leq k \leq n$,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. For the base case, we note that there is exactly one way of choosing 0 elements from an n element set, hence $\binom{n}{0} = 1$ and there is only one way of choosing n elements from an n element set so $\binom{n}{n} = 1$. In particular, $\binom{0}{0} = \frac{0!}{0!0!} = 1$, $\binom{1}{0} = \frac{1!}{0!1!} = 1$ and $\binom{1}{1} = \frac{1!}{1!0!} = 1$ so the statement we are trying to prove holds for $n = 0$ and $n = 1$.

Now assume that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for some fixed n and all $0 \leq k \leq n$. Then we have that for $1 \leq k \leq n$,

$$\begin{aligned}
\binom{n+1}{k} &= \binom{n}{k-1} + \binom{n}{k} \text{ (by Lemma 2)} \\
&= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \text{ (by the inductive hypothesis)} \\
&= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right) \text{ (algebra)} \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n+1}{k(n-k+1)} \text{ (more algebra)} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \text{ (algebra wins!)}
\end{aligned}$$

The case of $k = 0$ and $k = n + 1$ hold because of what appears in the first paragraph. Therefore by the P.M.I., $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $0 \leq k \leq n$ for all $n \geq 0$. $\square$

I showed at least one place where a combinatorial explanation is an incredibly powerful tool.

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}$$

In the case of $n = 0, 1, 2$ it says that

$$\binom{0}{0}^2 = 1 = \binom{0}{0}\binom{0}{0}\binom{0}{0}$$

$$\binom{2}{1}^2 = 4 = \binom{2}{0}\binom{2}{0}\binom{2}{1} + \binom{2}{1}\binom{1}{1}\binom{0}{0} = 2 + 2$$

$$\binom{4}{2}^2 = 36 = \binom{4}{0}\binom{4}{0}\binom{4}{2} + \binom{4}{1}\binom{3}{1}\binom{2}{1} + \binom{4}{2}\binom{2}{2}\binom{0}{0} = 6 + 24 + 6$$

Proof. The left hand side of this equation, $\binom{2n}{n}^2$, represents the number of ways of taking an urn with $2n$ labeled balls and choosing n of them to color red, then putting all of the balls back into the urn and reaching in and pulling out n of them to color blue.

Some of the balls that go through this procedure will have red and blue paint will be purple. Say that there are k balls which are colored red, then there will be k balls which are colored blue, and $n - k$ balls which are colored purple. The number of ways that this can happen is the same as the number of ways of reaching in the urn and pulling out k

balls to color red (in $\binom{2n}{k}$ ways), then from the remaining $2n-k$ balls picking k more to be blue (in $\binom{2n-k}{k}$ ways) and then picking $n-k$ from the remaining $2n-2k$ (in $\binom{2n-2k}{n-k}$ ways). By the multiplication principle there are $\binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}$ ways of having k red, k blue and $n-k$ purple balls with the rest unpainted. Since k can be any value between 0 and n, by the addition principle we have that the total number of ways of picking the balls and coloring them this way is

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k} \ .$$

□

We considered rearranging letters of a word. I looked at the number of rearrangements of the word ANNOTATE. Consider rearrangements of the letters like TNTAAOEN or NEONATAT. I said that the following procedure will determine the word

- pick two positions from 8 for the letter A
- pick one position from the remaining 6 for the letter E
- pick two positions from the remaining 5 for the letter N
- pick one position from the remaining 3 for the letter O

the remaining two positions of the word will be filled with T's. That the set of rearrangements of the word ANNOTATE is in bijection with the sequences of subsets of $\{1, 2, \ldots, 8\}$ consisting of a subset of size 2, a subset of size 1, a subset of size 2 and a subset of size 1.

For example the word TNTAAOEN is sent under this bijection to $(\{4,5\}, \{7\}, \{2,8\}, \{6\})$. The number of such sequences is equal to

$$\binom{8}{2}\binom{6}{1}\binom{5}{2}\binom{3}{1} = \frac{8!}{2!6!}\frac{6!}{1!5!}\frac{5!}{2!3!}\frac{3!}{1!2!} = \frac{8!}{2!1!2!1!}$$

For this we define the notation we will call the multi-choose or multinomial coefficient. We will define $\binom{n}{k_1,k_2,\cdots,k_r}$ to be the number of ways of picking subsets of size $k_1, k_2 \ldots, k_r$ from an n element set For a sequence of integers $k_1, k_2, \ldots, k_r \geq 0$ such that $k_1 + k_2 + \cdots + k_r \leq n$, then

$$\binom{n}{k_1, k_2, \cdots, k_r} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{r-1}}{k_r}$$
$$= \frac{n!}{k_1!k_2!\cdots k_r!(n-k_1-k_2-\cdots-k_r)!} \ .$$

If $k_1 + k_2 + \cdots + k_r > n$ then $\binom{n}{k_1,k_2,\cdots,k_r} = 0$.

There is another place where this coefficient arises. I assume that everyone is familiar with the binomial theorem which gives an expansion of $(1+x)^n$ in terms of the binomial coefficients $\binom{n}{k}$. We have

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$$

for example, we have in particular

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 + 0x^5 + 0x^6 + 0x^7 + \ldots$$

The multinomial coefficient is a generalization of these coefficients. In fact, we have

$$(1 + x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r \leq n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}$$

With so many unknowns in this equation it is hard to appreciate this formula. But try an example. I can use the computer and see that $(1 + x + y)^4 =$

$$1+4\,x+4\,y+6\,x^2+12\,xy+6\,y^2+4\,x^3+12\,x^2y+12\,xy^2+4\,y^3+x^4+4\,x^3y+6\,x^2y^2+4\,xy^3+y^4$$

I can use this formula to see that $\binom{4}{1,2} = \frac{4!}{1!2!1!} = 12$ and I see that the coefficient of xy^2 in this expression is 12. If I want to answer a question like what is the coefficient of $x^7y^3z^9$ in the expression $(1+x+y+z)^{40}$ then I have a formula for this value, it is $\binom{40}{7,3,9} = \frac{40!}{7!3!9!21!}$ just as the binomial theorem tells me the coefficient of x^{19} in $(1 + x)^{40}$ is $\binom{40}{19} = \frac{40!}{19!21!}$.

Remark 1: How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + \cdots + x_r = n?$$

Answer: $\binom{n+r-1}{n} = \binom{n+r-1}{r-1}$. Why? Think of a dots and bars argument and find a bijection from a solution to this equation represented as a sequence $(x_1, x_2, x_3, \ldots, x_r)$ and a sequence of n dots and $r - 1$ bars.

Remark 2: How many paths are there in a lattice grid from $(0, 0)$ to (n, m) with n steps $E = (1, 0)$ and m steps $N = (0, 1)$?

Answer: $\binom{n+m}{n} = \binom{n+m}{m}$. Why? Think of a lattice path in a grid with N and E steps and translate it into a word of letters N and E such that there are m letters N and n letters E. The number of such words is determined by the number of ways of choosing the positions of the E steps in the word.