TWENTY-THRID LECTURE : NOVEMBER 27, 2014

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I started with the problem "How many ways are there of putting 8 beads on a necklace using k colors assuming that adjacent beads cannot be the same color."

I found out that it is more difficult to count the colorings of necklaces with 8 beads with adjacent beads different colors than I thought and there was one step that I had wrong in my calculation that I did before class. I will put off the solution to this problem next time because I wanted to find a clever way to explain how to count these (the long way would have taken a long time).

The definition of a *partially ordered set* is a set S with a relation $\leq$ which has the properties

(1) reflexive - $x \leq x$ for $x \in S$
(2) transitive - if $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in S$
(3) anti-symmetric - for $x, y \in S$, if $x \leq y$ and $y \leq x$, then $x = y$ (this can also be stated as if x is less than y, then y is not less than x).

If x is not less than or equal to y and y is not less than or equal to x then we say that x and y are not comparable. If is always the case that either x is less than or equal to y or y is less than or equal to x for every x and y in the set then we say that the partial order is a *total order*.

Almost any set where you order the elements in some way is a partial order. If you have a way of deciding how every element compares to every other element then this is called a total order on a set (and it is still a partial order). However in most partial orders there are some elements which are not comparable.

Example 1. The first simple example that we are aware of just the positive integers with the relation $x \leq y$ representing "x is less than or equal to y." This is an example of a partial order which is a total order.

Example 2. The second example is the set of positive integers where the relation is "x divides y". We usually denote this relation $a|b$ to mean that there exists a k such that $ak = b$. Note that $a|b$ is either true or false, it is not a number. Any positive integer divides itself (so this relation is reflexive), if x divides y and y divides z then x divides z (so it is transitive). Also if x divides y and y divides x then x and y are equal (anti-symmetric).

This is an example of a partial order which is not a total order because for instance 3 does not divide 5 and 5 does not divide 3 so they are not comparable in this partial order.

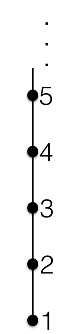
Example 3. The third example we take as our set the set of subsets of $\{1, 2, \ldots, n\}$ where n is a fixed integer (which is often denoted B_n). This example is slightly different than the previous two since for a fixed n it has a finite number of elements. A subset S will be less than a subset T if S is contained in T. Again we can check that this relation is reflexive, transitive and anti-symmetric. Since the empty set is a subset of every subset it is a smallest element of the set.

This is another example of a partial order which is not a total order because, for instance, the sets $\{1, 2\}$ and $\{2, 4, 6\}$ would not be comparable.

Example 4. As a fourth example I considered the set partitions of $\{1, 2, \ldots, n\}$ where n is again a fixed integer. We say that a set partition $\{S_1, S_2, \ldots, S_k\}$ is smaller than or equal to a set partition $\{T_1, T_2, \ldots, T_\ell\}$ if each set S_i is a subset a set T_j.

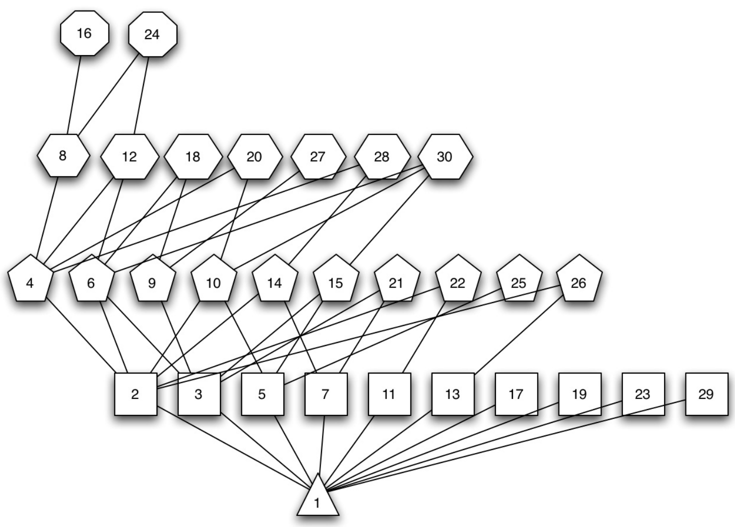
These partial orders can be pictured by drawing a dot for each element of the set and placing the smallest elements at the bottom of the picture and the larger elements above. Then we draw a line between x and y if $x \leq y$ and if there are no other elements z such that $x \leq z \leq y$. This is called the Hasse diagram of the poset.

Example 5. The diagram for the first few integers ordered by the usual less than or equal to order looks like a vertical line.



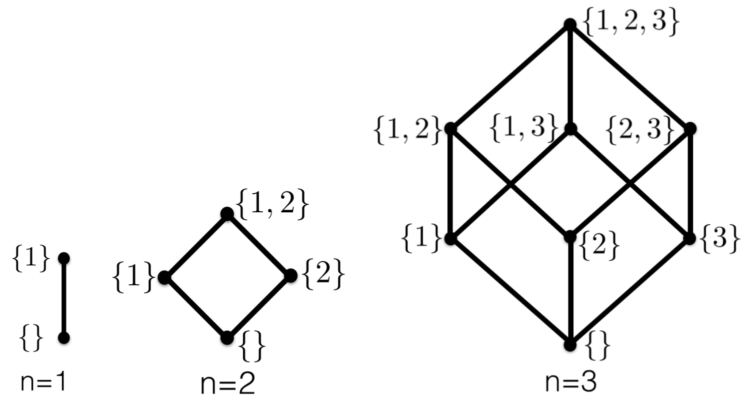
Example 6. The diagram for the first 30 integers ordered by division is the diagram

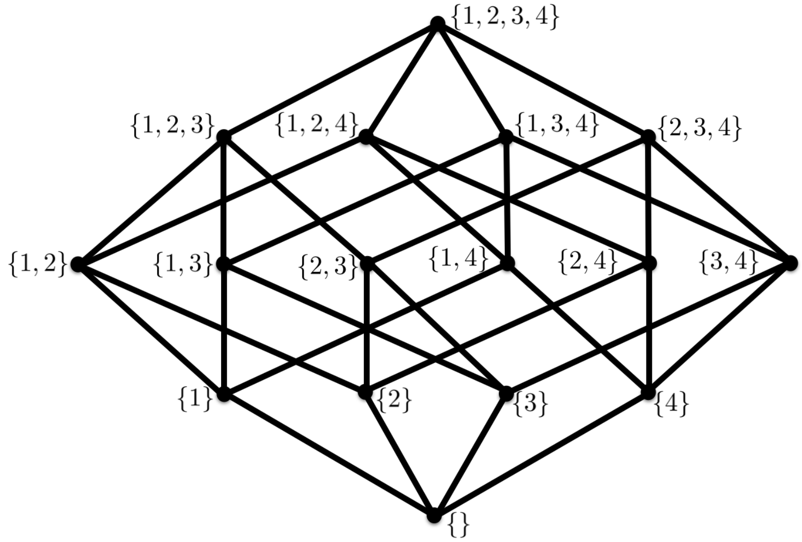
Graph of the integers 1-30 partially ordered by division



Notice that 1 is on the first level, the second level consists of the prime numbers, the third row consists of integers which are the product of two primes, the fourth level is the product of three primes, etc.

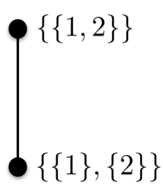
Example 7. The diagram for the poset of subsets of $\{1, 2, \ldots, n\}$ has the following diagrams for $n = 1, 2, 3, 4$.



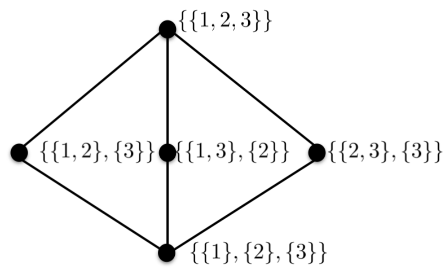


n=4

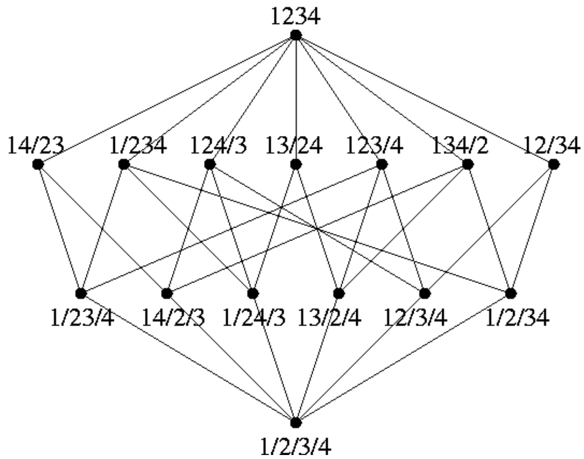
Example 8. For the partial order of set partitions defined in example 3 for $n = 2, 3, 4$ (for $n = 1$ there is just one element) the diagram for this partial order is the following three pictures.



n=2



n=3



$n = 4$

Note that the last image was taken from
`http://math.berkeley.edu/~lpachter/249_Spring_2009/reading/index.html`.

The set partitions are represented with the numbers that are in the same set in a group and the / indicates that the the next numbers are in a different set so that for instance the set partition $\{\{1,4\},\{2,3\}\}$ is represented by 14/23. The order of the groups and the numbers within the groups don't really matter.

With some minimal conditions on a partially ordered set we have the following theorem.

Theorem 9. *Let $(S, \leq)$ be a partially ordered set such that the number of elements less than or equal to any element x is finite, then given sequences a_x and b_x which are real numbers for each $x \in S$, there exists a real valued function μ on $S \times S$ such that*

$$a_x = \sum_{y \leq x} b_y$$

if and only if

$$b_x = \sum_{y \leq x} \mu(x,y) a_y.$$

Now for each poset it is an interesting problem to find what the function $\mu(x, y)$ is. I gave you the formula for three of these functions μ. One of them we had seen before.

Proposition 10. *(Telescoping sums) For n an integer,*

$$a_n = \sum_{i=1}^{n} b_i,$$

if and only if

$$b_n = a_n - a_{n-1} \ .$$

that is,

$$\mu(n,k) = \begin{cases} 1 & \text{if } k = n \\ -1 & \text{if } k = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 11. *(usual Möbius inversion) For n an integer,*

$$a_n = \sum_{d|n} b_d,$$

if and only if

$$b_n = \sum_{d|n} \mu(n,d) a_d$$

where,

$$\mu(n,d) = \begin{cases} (-1)^k & \text{if } n/d = p_1 p_2 \cdots p_k \text{ where all the } p_i \text{ are distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 12. *(Inclusion-Exclusion) For S a subset of the integers $\{1, 2, \ldots, n\}$,*

$$a_S = \sum_{T \subseteq S} b_T,$$

if and only if

$$b_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} a_T,$$

that is,

$$\mu(S,T) = \begin{cases} (-1)^{|S|-|T|} & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

When I mentioned that we had already seen one of these I was referring to Proposition 10. We used this in the first few days of class and it is called the method of telescoping sums. Look back at the notes from some of the first classes from this year.

____________________ From September 11, 2014 ____________________

In order to show that

$$b(1) + b(2) + \cdots + b(n) = a(n)$$

for some formulas $a(n)$ and $b(n)$ and $b(0) = 0$, then all you need to do is show that $a(n) - a(n-1) = b(n)$. If you do then

$$a(n) - a(n-1) = b(n)$$
$$a(n-1) - a(n-2) = b(n-1)$$
$$\vdots$$
$$a(2) - a(1) = b(2)$$

$$a(1) - a(0) = b(1)$$

Now add up all the terms on the left hand side and we have

$$(a(n) - a(n-1)) + (a(n-1) - a(n-2)) + \cdots + (a(1) - a(0)) = a(n) - a(0) = a(n)$$

If you add up all the terms on the right hand side of the equality then you have

$$b(n) + b(n-1) + \cdots + b(2) + b(1)$$

and they must be equal.

(For point of clarification, I exchanged a and b in this excerpt to ensure that they agree my notation here)

Lets do the same thing for the Proposition 11. In this case the first 8 terms of our sequence are

$$a_1 = b_1$$
$$a_2 = b_2 + b_1$$
$$a_3 = b_3 + b_1$$
$$a_4 = b_4 + b_2 + b_1$$
$$a_5 = b_5 + b_1$$
$$a_6 = b_6 + b_3 + b_2 + b_1$$
$$a_7 = b_7 + b_1$$
$$a_8 = b_8 + b_4 + b_2 + b_1$$
$$\vdots$$

Now if we start by solving for the b-sequence in terms of the a-sequence the we see

$$b_1 = a_1$$
$$b_2 = a_2 - a_1$$
$$b_3 = a_3 - a_1$$
$$b_4 = a_4 - a_2$$
$$b_5 = a_5 - a_1$$
$$b_6 = a_6 - a_3 - a_2 - a_1$$
$$b_7 = a_7 - a_1$$
$$b_8 = a_8 - a_4$$
$$\vdots$$

It looks the same as in the case of $a_n = \sum_{i=1}^{n} b_n$ except that in the case of b_6 there are 4 terms. It is rarely the case the case that there is just one exception. In fact, Proposition

11 is telling us that there will be 2^k terms in the expression for b_n if n is a product of k distinct primes. For example if we solve for b_{36}, because $36 = 2^2 \cdot 3^2$ we have that

$$b_{36} = a_{36} - a_{18} - a_{12} + a_6$$

while since $30 = 2 \cdot 3 \cdot 5$ we will have

$$b_{30} = a_{30} - a_{15} - a_{10} - a_6 + a_5 + a_3 + a_2 - a_1 \ .$$

For each of these propositions, there is something to prove. We have more or less shown Proposition 10. The ones that are slightly more difficult to show are Propositions 11 and 12.

1. Inclusion-Exclusion

Proposition 12 is the easier of the two to prove (I will include the proofs below) and it can be used to solve a class of counting problems that are elementary but difficult (remember the definition of 'elementary' is that it is easy to state with very little background knowledge, not that it is easy to solve). Before I provide the justification, let me show you the application.

Example 13. I went to a textbook on combinatorics and looked that there was a whole section of problems on inclusion-exclusion. I just picked out the first problem I could find:

How many 5 card hands have at least one card from each suit?

Let S be a set of suits

$A_S =$ # of 5 card hands with suits from S (some suits might not be included)

and

$B_S =$ # of 5 card hands with suits exactly from S (each suit appears at least once)

then these sequences of numbers are related by

$$A_S = \sum_{T \subseteq S} B_T$$

because of the addition principle. Every 5 card hand has a unique set of suits which appear in the hand T and that hand will be counted in the count from B_T. That is not true hence (by Möbius inversion/inclusion-exclusion) they are also related by

$$B_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} A_T \ .$$

The thing is we can count the number of hands in A_S. If $S = \{\heartsuit\}$ or $S = \{\clubsuit\}$ or $S = \{\spadesuit\}$ or $S = \{\diamondsuit\}$, then $A_S = A_{\{\heartsuit\}} = \binom{13}{5}$. If S has two suits (e.g. $S = \{\heartsuit, \spadesuit\}$ and there are 6 sets like this) then $A_S = \binom{26}{5}$. If S contains three suits (e.g. $S = \{\heartsuit, \spadesuit, \diamondsuit\}$

and there are 4 sets like this) then $A_S = \binom{39}{5}$. Finally $A_{\{\heartsuit,\spadesuit,\clubsuit,\diamondsuit\}} = \binom{52}{5}$. In case it needs to be said, $A_{\{\}} = 0$. Therefore,

$$B_{\{\heartsuit,\spadesuit,\clubsuit,\diamondsuit\}} = \binom{52}{5} - 4\binom{39}{5} + 6\binom{26}{5} - 4\binom{13}{5}$$

and this is equal to the number of 5 card hands which contain at least one card of each suit.