F. FRANKLIN'S PROOF OF EULER'S PENTAGONAL NUMBER THEOREM

Abstract. The 18^{th} century mathematician Leonard Euler discovered a simple formula for the expansion of the infinite product $\prod_{i\geq 1} 1-q^i$. In 1881, one of the first American mathematicians found an elegant combinatorial proof of this identity.

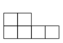
Proposition 1. *(Euler's pentagonal number theorem)*

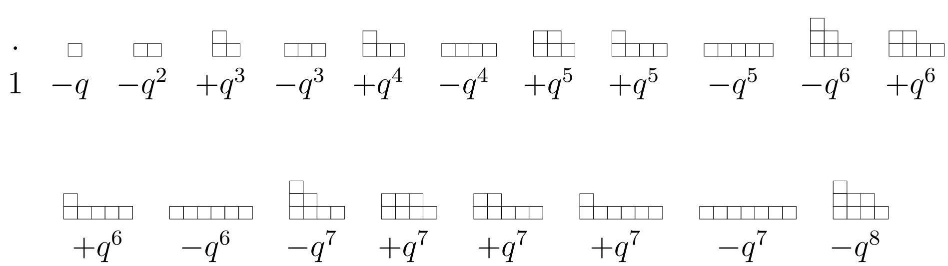
$$\prod_{i\geq 1} 1-q^i = 1+\sum_{m\geq 1}(-1)^m\left(q^{\frac{m(3m-1)}{2}}+q^{\frac{m(3m+1)}{2}}\right) \tag{1}$$

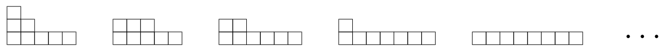
There is a clever proof of this proposition that comes from a mathematician F. Franklin [4]. Since this is exactly the sort of proof that is in the spirit of mathematics of algebraic combinatorics it belongs in a course on algebraic combinatorics. Other accounts of this proof can be found in: [5], [6], [7], [8], [9].

Example 1. We note that the left hand side of this equation is the generating function for all strict partitions (partitions where all parts are distinct) weighted with $(-1)^{\ell(\lambda)}q^{|\lambda|}$. That is,

$$\prod_{i\geq 1} 1-q^i = \sum_{\lambda\ strict}(-1)^{\ell(\lambda)}q^{|\lambda|} \tag{2}$$

This follows by observing that to determine the coefficient of q^n by expansion of the product on the left we have a contribution of $(-1)^k q^{\lambda_1+\lambda_2+\cdots+\lambda_k}$ for every sequence $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ such that $\lambda_i>\lambda_{i+1}$ for $1\leq i<k$. Below we expand the terms of this generating function through degree 8. For example, a term of the form $(-q^4)(-q^2)$ is represented by the picture  and we record the weight of $+q^6$ just below the picture.



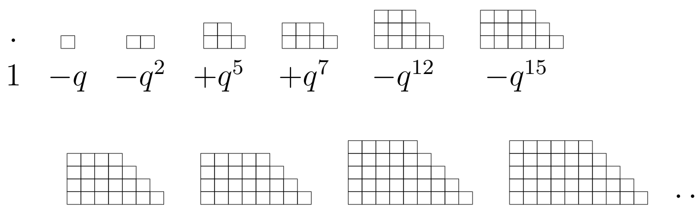


$-q^8 \quad +q^8 \quad +q^8 \quad +q^8 \quad -q^8 \quad + \cdots$

Now we notice that all of the terms cancel except for the ones stated in the theorem, that is we have

$$\prod_{i \geq 1} 1 - q^i = 1 - q - q^2 + q^5 + q^7 + \cdots$$

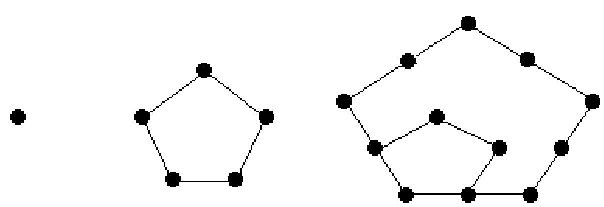
In fact, we will show that one way of looking at this expression is to observe terms which survive are those that correspond to the following pictures:



$\prod_{i\geq 1} 1 - q^i = \quad 1 \quad -q \quad -q^2 \quad +q^5 \quad +q^7 \quad -q^{12} \quad -q^{15}$

$+q^{22} \quad +q^{26} \quad -q^{35} \quad -q^{40} \quad \cdots$

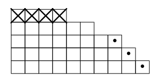
From the image in this example one might think that the theorem would be better named the *trapazoidal* number theorem. There is a reason that the numbers $m(3m-1)/2$ are referred to as pentagonal numbers and if $m \to -m$ then the pentagonal number is transformed to $\to -m(-3m-1)/2 = m(3m+1)/2$. Observe the picture below how a sequence of pentagons have exactly $m(3m-1)/2$ points in them (and this continues for $m > 3$).



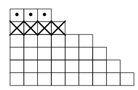
Proof. To show that this proposition holds we show that there is an involution ϕ on the strict partitions λ of n such that $\phi(\lambda)$ is also a partition of n and the length of $\phi(\lambda)$ will have length either one smaller or one larger than that of λ. This means that if the weight of a strict partition is $(-1)^{\ell(\lambda)} q^{|\lambda|}$ then the weight of $\phi(\lambda)$ is $-(-1)^{\ell(\lambda)} q^{|\lambda|}$ and so this term corresponding to $\phi(\lambda)$ will cancel with the term corresponding to λ. This involution will fail to 'work' for the partitions of the form $(2m-1, 2m-2, \ldots, m)$ which are of size $2m^2 - \frac{(m+1)m}{2} = \frac{m(3m-1)}{2}$ and $(2m, 2m-1, \ldots, m+1)$ which are of size $2m^2 - \frac{(m-1)m}{2} = \frac{m(3m+1)}{2}$.

For a strict partition λ we will let r equal to the smallest part of λ ($r = \lambda_{\ell(\lambda)}$) and let s equal the number of parts which are consecutive at the beginning of the partition. In other words s is the largest integer such that $(\lambda_1, \lambda_2, \ldots, \lambda_s) = (\lambda_1, \lambda_1 - 1, \ldots, \lambda_1 - s + 1)$.

If $s \neq \ell(\lambda)$ and $r > s$ then we will let $\phi(\lambda)$ equal the partition $(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_{\ell(\lambda)}, s)$. That is, if the diagram for the partition looks something like the following where there is an $\times$ in each of the cells corresponding to r and a dot in the cells corresponding to s

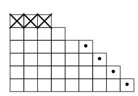


then $\phi(\lambda)$ will be the partition with the diagonal of s cells filled with a dot moved to the top row of the partition.

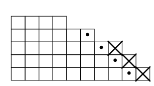


$\phi(\lambda)$ has the property that the longest string of consecutive parts at the beginning of the partition is greater than or equal to s.

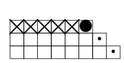
If $s \neq \ell(\lambda)$ and $r \leq s$ then we will let $\phi(\lambda)$ equal to the partition $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{\ell(\lambda)})$. For example, if our diagram is similar to the one below with the cells marked with an $\times$ representing the row of size r and those marked with the $\cdot$ represent the cells which correspond to the s consecutive parts at the beginning of the partition.



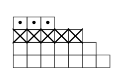
The partition corresponding to $\phi(\lambda)$ is then represented by the following picture.



Notice that it is also possible that $s = \ell(\lambda)$. In this case if $r > s + 1$ then we will remove the s cells along the diagonal and turn them into the shortest row so that $\phi(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, s)$. For example we have the picture on the left will be transformed to the one on the right.

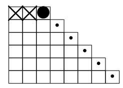


λ

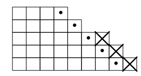


$\phi(\lambda)$

If $s = \ell(\lambda)$ and $r < s$ then we will set $\phi(\lambda) = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_r, \ldots_{\ell(\lambda)-1})$, this corresponds to the case when we have a partition of the form of the one below.



λ



$\phi(\lambda)$

If we describe what is happening to the diagram the map ϕ does one of two things, either it removes the smallest row of $r = \lambda_{\ell(\lambda)}$ cells of the partition and places one cell more in each of the first r rows (in the case that $r < s$ or $r = s$ and $s < \ell(\lambda)$) or it removes one cell from each of the first s rows and adds a row of size s to the top of the diagram (in the case that $r > s + 1$ or $r = s + 1$ and $s < \ell(\lambda)$).

Observe that if the weight of λ is $(-1)^{\ell(\lambda)}$ then since $\phi(\lambda)$ has the same number of cells and either one more or one less row than λ then the weight of $\phi(\lambda)$ is the negative of the weight of λ.

Also observe for each of the 4 cases we have considered, $\phi(\phi(\lambda))$ is just λ. This implies we can say that in the expansion of the expression $\sum_{\lambda\ strict}(-1)^{\ell(\lambda)}q^{|\lambda|}$, the term corresponding to the partition λ will cancel with the term corresponding to the partition $\phi(\lambda)$.

There are two cases that we have not considered. These terms do not cancel. One is that $r = s$ and $s = \ell(\lambda)$ and so we have a partition of the form $(2m-1, 2m-2, \ldots, m)$ and the other is that $r = s+1$ and $s = \ell(\lambda)$ and this is a partition of the form $(2m, 2m-1, \ldots, m+1)$. □

We encourage the reader to take a pencil and draw an arrow between the diagrams of the strict partitions given in the example above to show that the involution works as expected.

References

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