MIDTERM # 1 - MATH 4160 - JANUARY 31, 2003

SOLUTION TO PROBLEM #4

Prove one of the following identities by giving a combinatorial interpretation to both sides of the equation and explaining why they must be equal:

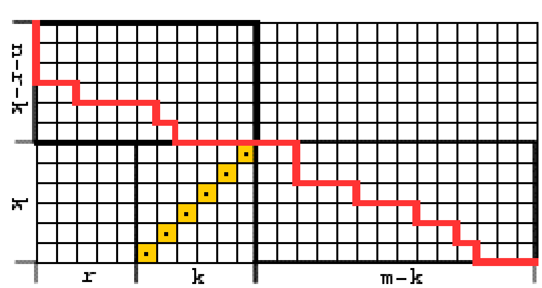
$$\binom{m+n}{m+r} = \sum_{k=0}^{m} \binom{m}{m-k}\binom{n}{r+k}$$

Proof 1: The left hand side of this equation counts the number of ways of choosing $m+r$ balls from a set of $m+n$ numbered balls labeled 1 through $m+n$ and coloring those $m+r$ balls red.

Fix a k between 0 and m. $\binom{m}{m-k}\binom{n}{r+k}$ counts the number of ways of choosing $m-k$ balls from the first m and coloring them red and then choosing $r+k$ from the n balls labeled $m+1$ through $m+n$ and coloring those red. In total there will be $m-k+r+k = m+r$ balls colored red. Every time I pick $m+r$ balls to color red $m-k$ of them will be labeled 1 through m where $0 \leq k \leq m$. These are disjoint outcomes so the right hand side of this equation also counts the number of ways of choosing $m+r$ balls from a set of size $m+n$ numbered balls to color red.

Proof 2: Fix the integers r, m and n. If $r \leq n$, then $\binom{m+n}{m+r}$ is non-zero and this quantity represents the number of lattice paths from the point $(0, n-r)$ to $(m+r, 0)$ which have south steps $(0,-1)$ and east steps $(1,0)$.

For a lattice path in the $(m+r) \times (n-r)$ rectangle, let k be the size of the largest square which lies to the right of the line $x = r$ and underneath the path. This defines a rectangle which is of size $(r+k) \times k$ which fits underneath the path. Every path of this type passes through the point $(r+k, k)$.



Now the number of paths which pass through the point $(r+k,k)$ is equal to the number of paths which pass travel from $(0,n-r)$ to $(r+k,k)$ (there are $\binom{n}{r+k}$ such paths) times the number of paths which go from the point $(r+k,k)$ to $(m+r,0)$ (there are $\binom{m}{m-k}$ such paths).

Therefore the number of partitions with k equal to the largest square that lies to the right of the line $x=r$ and fits underneath the path is equal to $\binom{m}{m-k}\binom{n}{r+k}$.

Since every path from $(0,n-r)$ to $(m+r,0)$ has a unique value of k which is the largest square that lies to the right of the line $x=r$ and fits underneath the path and this value of k ranges from 0 to m the right hand side of this equation also counts the number of lattice paths from $(0,n-r)$ to $(m+r,0)$ and hence both sides of this equation are equal.

$$\binom{n}{k}\binom{k}{\ell} = \binom{n-1}{k}\binom{k}{\ell} + \binom{n-1}{k-1}\binom{k-1}{\ell} + \binom{n-1}{k-1}\binom{k-1}{\ell-1}$$

Proof 1: The left hand side of this equation counts the number of ways of separating k balls from a set of n numbered balls and coloring the remaining $n-k$ balls green, then choosing from the k separated balls ℓ to be colored red, and coloring the remaining $k-\ell$ balls blue. This counts the number of partitions of a set of n numbered elements into three subsets: a green subset of size $n-k$, a red subset of size ℓ, and a blue subset of size $k-\ell$.

Assume that the ball labeled with an n is green. The number of partitions of the remaining balls into subsets of these sizes is equal to the number of ways of choosing k elements from the remaining $n-1$ to be colored red and blue and coloring the remaining $n-k-1$ green and then choosing ℓ of the subset of size k balls to be red and coloring the remaining $k-\ell$ balls blue. This can be done in $\binom{n-1}{k}\binom{k}{\ell}$ ways.

Assume that the ball labeled with an n is blue. The number of ways of partitioning the set into a red, blue and green set is equal to the number of ways of choosing $k-1$ elements from the $n-1$, coloring the remaining $n-k$ balls green, then choosing ℓ balls to be colored red, and coloring the remaining $k-1-\ell$ balls blue. This can be done in $\binom{n-1}{k-1}\binom{k-1}{\ell}$ ways.

Assume that the ball labeled with n is red. The number of ways of partitioning the set into a red, blue and green set is equal to the number of ways of choosing $k-1$ elements from the $n-1$ to be red and blue, coloring the remaining $n-k$ elements green, then choosing $\ell-1$ of the $k-1$ balls to be colored red and finally coloring the remaining $k-\ell$ balls blue. This can be done in $\binom{n-1}{k-1}\binom{k-1}{\ell-1}$ different ways.

Since the right hand side of this equation counts the number of ways of partitioning the set of n balls into three colored sets with $n-k$ green balls, $k-\ell$ blue balls and ℓ red balls with either the ball labeled n colored green or colored blue or colored red, the two sides of this equation must be equal.

Proof 2: I don't think that this identity lends itself well to a combinatorial interpretation in terms of lattice paths.