MIDTERM # 2 - SOLUTIONS - MATH 4160

ASSIGNED: MARCH 7, 2003 DUE: MARCH 10, 2003 AT 10:30AM

Part (b) of problem #4 and #5 are questions that were only asked of the grad students in this course.

(1) Say that there are exactly 100 unlabeled balls in a bin but that they are colored so that 25 are red, 25 are green, 25 are blue and 25 are yellow. Reach in the bin and pick out 4 balls. What is the probability of choosing them such that 2 balls are one color and 2 balls are a second color?

Answer: The number of ways of picking 4 balls so that 2 are of one color and 2 are another is the number of ways of specifying the two colors $\binom{4}{2}$ then picking 2 balls from the first color and 2 balls from the second color. The number of ways of picking 4 balls from a bin of 100 balls is $\binom{100}{4}$. Therefore the probability of arriving at an outcome with 2 of one color and 2 of another is

$$\frac{\binom{4}{2}\binom{25}{2}\binom{25}{2}}{\binom{100}{4}} = \frac{7200}{52283} \approx .1377120670.$$

(2) Consider paths in a 3-dimensional grid starting at the origin $(0,0,0)$ and ending at the point (n,k,ℓ).
   (a) How many paths are there if there are only steps in the $(+1,0,0)$, $(0,+1,0)$, or $(0,0,+1)$ direction?
   (b) Write down a formula for the number of paths there are if in addition to steps in the $(+1,0,0)$, $(0,+1,0)$ or $(0,0,+1)$ directions that there are also steps in the $(+1,+1,+1)$ direction. Hint: grade your answer by the number of diagonal steps.

Answer (a): A path of this type corresponds to a word with $n+k+\ell$ letters with n A's, k B's, and ℓ C's. The number of words of this type are

$$\binom{n+k+\ell}{n}\binom{k+\ell}{k}.$$

Answer (b): A path with d diagonal steps corresponds to a word with $n+k+\ell-2d$ letters with $n-d$ A's, $k-d$ B's, $\ell-d$ C's, and d D's. The number of words of this type are

$$\binom{n+k+\ell-2d}{n-d}\binom{k+\ell-d}{k-d}\binom{\ell}{\ell-d} = \binom{n+k+\ell-2d}{d}\binom{n+k+\ell-3d}{n-d}\binom{k+\ell-2d}{k-d}.$$

Since d may be any number between 0 and the minimum of n, k and ℓ and the set of paths with d diagonal steps is disjoint from the the set of paths with d' diagonal steps if $d \neq d'$

then we have that the total number of steps is

$$\sum_{d=0}^{min(n,k,l)} \binom{n+k+\ell-2d}{n-d}\binom{k+\ell-d}{k-d}\binom{\ell}{\ell-d}.$$

(3) Give a formula for the generating function of the number of partitions satisfying there is exactly one even part and every odd part is not repeated more than twice (e.g. $(4,3,1,1)$ is a partition of this type but $(4,3,1,1,1)$ is not).

Answer: The generating function for a part of size 2 or 4 or 6 etc. is $\frac{x^2}{1-x^2}$. The generating function for either 0 or 1 or 2 parts of size k is $1+x^k+x^{2k} = \frac{1-x^{3k}}{1-x^k}$. The partitions that we are considering is a concatenation of one even part and $0,1$ or 2 parts of size $2k+1$ for all $k \geq 0$. That is the generating function we are looking for is:

$$\frac{x^2}{1-x^2}\prod_{k\geq 0}\frac{1-x^{6k+3}}{1-x^{2k+1}} = \frac{x^2}{1-x^2}\frac{(1-x^3)(1-x^9)(1-x^{15})(1-x^{21})\cdots}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots}$$

$$= \frac{x^2}{1-x^2}\prod_{k\geq 0}\frac{1}{1-x^{6k+1}}\frac{1}{1-x^{6k+5}}$$

(4) (a) Say that a_i is the number of *widgets* of size i and let $A(x) = \sum_{n\geq 0} a_n x^n$ be the generating function for these numbers. What is the coefficient of x^n in the expression $(1+x)^m A(x) = \sum_{k=0}^{m}\binom{m}{k}x^k A(x)$? Give an explanation of the meaning of this coefficient in terms of *widgets*.

Answer:

$$\text{the coefficient of } x^n = \sum_{k=0}^{m}\binom{m}{k}a_{n-k}$$

One 'meaning' of this coefficient is it counts the set of pairs (S,w) where S is a subset of $\{1,2,\ldots,m\}$ and w is a widget of size $n-|S|$.

(b) Let $B(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ be the exponential generating function for these numbers. What is the coefficient of $x^n/n!$ in the expression $(1+x)^m B(x) = \sum_{k=0}^{m}\binom{m}{k}x^k B(x)$? Give an explanation of the meaning of this coefficient in terms of *widgets*.

$$(1+x)^m B(x) = \sum_{r\geq 0}\sum_{k=0}^{m}\binom{m}{k}a_r x^k \frac{x^r}{r!} = \sum_{r\geq 0}\sum_{k=0}^{m}\binom{m}{k}a_r \frac{(r+k)!}{r!}\frac{x^{r+k}}{(r+k)!}$$

Now the coefficient of $\frac{x^n}{n!}$ is

$$\sum_{k=0}^{m}\binom{m}{k}\frac{n!}{(n-k)!}a_{n-k}.$$

This coefficient then counts the number of triples (S,π,w) where S is a subset of the integers $\{1,2,\ldots,m\}$, π is a sequence of integers $\pi_1\pi_2\cdots\pi_{|S|}$ where $\pi_i \neq \pi_j$ if $i \neq j$ and $1 \leq \pi_i \leq n$ and w is a widget of size $n-|S|$.

(5) Find a formula for the generating function $D(x) = \sum_{n\geq 0} d_n x^n$ for the sequence d_n that satisfies the recurrence $d_n = 2d_{n-1} - d_{n-2} + n$ and $d_0 = 1$ and $d_1 = 3$.

Answer:

$$\begin{aligned} D(x) &= \sum_{n\geq 0} d_n x^n = 1 + 3x + \sum_{n\geq 2}(2d_{n-1} - d_{n-2} + n)x^n \\ &= 1 + 3x + 2(xD(x) - x) - x^2 D(x) \sum_{n\geq 2} nx^n + x - x \\ &= 1 + 2xD(x) - x^2 D(x) + \frac{x}{(1-x)^2} \end{aligned}$$

Therefore

$$D(x) - 2xD(x) + x^2 D(x) = 1 + \frac{x}{(1-x)^2} = \frac{1 - x + x^2}{(1-x)^2}$$

and so

$$D(x) = \frac{1 - x + x^2}{(1-x)^4}.$$

(b) Find an expression for the numbers d_n which is not recursive by taking the coefficient of x^n in the expression for $D(x)$ that you just derived. Verify that this formula gives you the values of $d_1 = 3$, $d_2 = 7$, $d_3 = 14$, $d_4 = 25$.

Answer:

$$\begin{aligned} D(x) &= \frac{1-x}{(1-x)^4} + \frac{x^2}{(1-x)^4} = \frac{1}{(1-x)^3} + \frac{x^2}{(1-x)^4} \\ &= \sum_{n\geq 0} \binom{n+2}{2} x^n + \sum_{n\geq 0} \binom{n+1}{3} x^n \end{aligned}$$

and so $d_n = \binom{n+2}{2} + \binom{n+1}{3}$. This checks out since we observe that $d_1 = \binom{3}{2} + \binom{2}{3} = 3$, $d_2 = \binom{4}{2} + \binom{3}{3} = 6 + 1$, $d_3 = \binom{5}{2} + \binom{4}{3} = 10 + 4$, $d_4 = \binom{6}{2} + \binom{5}{3} = 15 + 10$.