TAKE HOME FINAL EXAM - GRAD VERSION
MATH 4160 - APRIL 4, 2003

You may use your book, notes or calculator. Do not discuss this exam with your fellow students.

Some notes: This exam is due on Friday April 18, 2003 at 11:30am. This due date is soft in the sense that if you NEED more time due to other constraints I will grant it to you but you must e-mail me before the due time to ask for it (`zabrocki@mathstat.yorku.ca`). The exam however should take no more than 3 hours if you have prepared for it and so I do not want to extend the due date by more than 3 days.

The course you have just completed covered three major areas of combinatorics: basic techniques of counting, generating functions and enumeration of symmetry classes of permutations (Pòlya enumeration). To attain an A in this course you must be able to answer questions in all three areas. Therefore I will break this exam into three smaller exams. Each one should take about an hour to complete (assuming that you are prepared) and each one covers roughly the topics that we discussed in a third of the course.

I will be available for office hours during the following times:

- Monday, April 7, 2003 1pm - 3pm
- Tuesday, April 15, 2003 12pm - 2pm

If you are unable to make either of these times and you would like to see me to ask questions regarding this exam feel free to e-mail me and we can try to arrange another time or if it is possible to answer your questions by e-mail I will try.

Basic Counting Techniques:

(1) (7 pts) How many ways are there of coloring n numbered balls with k colors?
(2) (7 pts) How many ways are there of coloring n numbered balls using k colors such that each color is used at most once?
(3) (7 pts) How many ways are there of coloring n indistinguishable balls using k colors?
(4) (7 pts) How many ways are there of coloring n indistinguishable balls with k colors such that each color is used at most once?
(5) (7 pts) How many distinct ways are there of rearranging the letters of the word BRIGHT SUNLIGHT (ignore the space)?

(6) (7 pts) How many distinct ways are there of rearranging the letters of the word ELECTRONICALLY such that the vowels are not next to each other?

(7) (7 pts) 60 days are chosen at random during the year. What is the probability that there are 5 days chosen from each month (assume non-leap year with 365 days; there are 30 days in September, April, June, and November, 28 in February, and 31 in the other 7 months)?

(8) (7 pts) If a 9 digit Social Insurance number is chosen at random what is the probability that at least two digits are the same?

(9) (7 pts) There are 50 cards numbered 1 through 50. Two different cards are chosen at random. What is the probability that one number is twice the other number?

(10) (8 pts) How many different positive integers can be obtained as a sum of two or more of the numbers 1, 2, 5, 10, 20, 37, 77?

(11) (9 pts) How many numbers less than 9500 are not divisible by 5, 7 and 13? (Hint: you will probably need to use Inclusion-Exclusion to solve this problem)

(12) (10 pts) How many sequences are there of 26 letters with each of the letters of the alphabet appearing exactly once and that do not contain the sequences of letters ABCD, ZYXW, MNOP, or SNOT? (Hint: you will probably need to use Inclusion-Exclusion to solve this problem)

(13) (10 pts) Give a combinatorial proof of the following identity:

$$\begin{bmatrix} n+r+s+2 \\ n \end{bmatrix} = \sum_{i=0}^{n} q^{ri+i} \begin{bmatrix} r+n-i \\ n-i \end{bmatrix} \sum_{j=0}^{i} q^{j} \begin{bmatrix} s+j \\ j \end{bmatrix}$$

Generating Functions, Partitions and Combinatorial Interpretations:

(1) (15 pts) Let a_k be the sequence defined by $a_0 = 1$ and $a_n = 2^n - a_{n-1}$. Give an expression for the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.

(2) (15 pts) Find a non-recursive expression for a_n.

(3) (20 pts) Let b_k be the sequence defined by $b_0 = 1$ and $b_n = 1 + b_{n-1} + b_{n-3} + b_{n-5} + \cdots$. Find an expression for the generations function $B(x) = \sum_{n\geq 0} b_n x^n$.

(4) (15 pts) Give an expression for the generating function for the number of partitions with only odd parts and parts of size k are repeated less than k times for all $k \geq 1$.

(5) (15 pts) Give an expression for the generating function the number of partitions with the property that every a part of size k either appears 0 or 2 times for all $k \geq 1$. (e.g. $(3,3,1,1)$, $(3,3)$, $(4,4,1,1)$, $(3,3,2,2,1,1)$ are all partition of this type and $(4,4,2,2,1)$ is not).

(6) (5 pts each) For each of the following expressions give a description of a set combinatorial objects which are enumerated by the formula.

(a) $2^n n!$

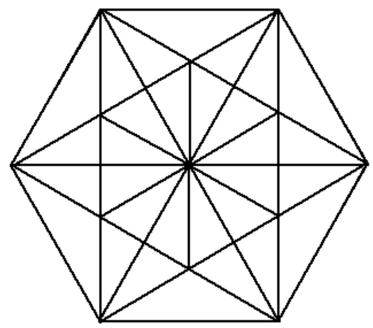
(b) $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots$

(c) $p(n) \cdot n!$ where $p(n)$ is the number of partitions of n.

(d) $\binom{n}{k} + n^k + k^n$

Burnside's Theorem and Pólya Enumeration:

The first 4 questions below relate to the diagram shown below which is invariant under rotations by 60° and a reflection about the vertical line. These two elements generate a dihedral group of order 12. Let $x =$ rotation by 60° which has has order 6, $x^6 = e$. Let $y =$ reflection about the vertical line which has order 2, $y^2 = e$. You may verify that $xy = yx^{-1}$.



(1) (15 pts) Consider the colorings of the 30 regions of the diagram which are distinct when we consider rotations of the diagram only. How many ways are there of coloring the regions with 5 colors?
(2) (15 pts) Redo the previous problem considering this time colorings of the diagram with 5 colors which are distinct under rotations and reflections of the diagram.
(3) (15 pts) Give the cycle index polynomial of the group of rotations and reflections acting on the group above.
(4) (20 pts) How many different patterns considering rotations and reflections are there when the regions of the diagram are colored so that 9 are red, 6 are green and 9 are blue and 6 are yellow.
(5) (20 pts) How many different patterns considering rotations only are there when the regions of the diagram are colored so that 9 are red, 6 are green and 9 are blue and 6 are yellow BUT the 6 larger triangles on the outside edge of the figure may not be colored yellow and can only be red, green or blue.
(6) (15 pts) The permutation $\pi \in S_4$ acts on a pair (a, b) with $a \neq b$ and $a, b \in \{1, 2, 3, 4\}$ by $\pi(a, b) = (\pi_a, \pi_b)$. Give the cycle index polynomial of the group action on the 12 pairs $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$.

Bonus: (10 pts) How many simple directed graphs are there with 4 unlabeled vertices and 7 edges?

Remark: A simple directed graph is a set of vertices V together with a set of edges E where E is a set of ordered pairs (v, w) with $v, w \in V$ and $v \neq w$. If V is the set of integers $\{1, 2, \ldots, n\}$, then the permutations in S_n act on the vertices and edges in a natural manner (as in problem #5 above). If we ask for the number of distinct simple directed graphs on n unlabeled vertices with k edges, then we are asking for the number of colorings of the set of edges with k of them black and the remaining $n(n-1) - k$ colored white that are distinct under the action of the symmetric group.