HOMEWORK #1 - MATH 4160

ASSIGNED: JAN 13/03, REVISED: JAN 16/03, DUE: JAN 24/03 AT 10:30AM

Enumeration problems:

(1) How many ways are there of placing two Kings on an 8×8 chessboard so that they are not on adjacent squares?

One King can be placed in either a corner, an edge, or in the interior of the chessboard. If it is on the corner then there are 60 other places to put the other King, on the edge there are 58 other places to place the second King, in the interior there are 55 places where the second King can go. If the Kings are different colors (first King black, second King white) then there are

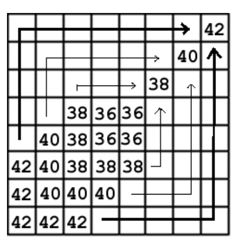
$$4 \cdot 60 + 24 \cdot 58 + 36 \cdot 55 = 3612$$

placements of two Kings. If the Kings are the same color (indistinguishable) then the number of placements is

$$(4 \cdot 60 + 24 \cdot 58 + 36 \cdot 55)/2 = 1806.$$

(2) How many ways are there of placing two Queens on an 8×8 chessboard so that the pieces are not on the same row, column or diagonal?

Below I have written in the number of cells that a second Queen could go if the first Queen was placed on that cell on the chessboard.



Therefore the number of ways of placing two queens on the chessboard if the first one is black and the second one is white is

$$28 \cdot 42 + 20 \cdot 40 + 12 \cdot 38 + 4 \cdot 36 = 2576.$$

If the two queens are the same color (indistinguishable) the number of ways is

$$(28 \cdot 42 + 20 \cdot 40 + 12 \cdot 38 + 4 \cdot 36)/2 = 1288.$$

(3) How many different r^{th} order partial derivatives does $f(x_1, x_2, \ldots, x_n)$.

Each partial derivative of order r is of the form

$$\partial_{x_1}^{a_1} \partial_{x_2}^{a_2} \cdots \partial_{x_n}^{a_n} f(x_1, x_2, \ldots, x_n)$$

with $a_1 + a_2 + \cdots + a_n = r$. The number of sequences $(a_1, a_2, \ldots, a_n)$ with $\sum_{i=1}^n a_i = r$ is equal to $\binom{n+r-1}{r}$ because every such sequence corresponds to a sequence of $\bullet$'s and $|$'s with a_1 $\bullet$'s followed by a $|$, followed by a_2 $\bullet$'s followed by a $|$, $\ldots$, and a_{n-1} $\bullet$'s followed by a $|$ followed by a_n $\bullet$'s. In total there are r $\bullet$'s (because the a_i's add up to r) and $n-1$ $|$'s (because there are n values of a_i) and every one of these sequences of $\bullet$'s and $|$'s corresponds to a sequence of integers $(a_1, a_2, \ldots, a_n)$ with $\sum_{i=1}^n a_i = r$. There are $\binom{n-1+r}{r}$ ways of placing r $\bullet$'s in a sequence of $n-1$ $|$'s and r $\bullet$'s ($n-1+r$ total) and the number of these is is the number of sequences of a_i.

(4) How many arrangements are there of five as, five bs and five cs with at least one b and at least one c between each successive pairs of a?

We are counting sequences of 15 letters and the as must be separated by at least 2 letters and there is at least one b in between each a and at least one c in between each a. I will list the patterns where the as can go grouped so that the number of ways of placing bs cs in that pattern of as is the same.

Group 1:

$a--a--a--a--a--$, $-a--a--a--a--a-$, $--a--a--a--a--a$

Group 2:

$a--a--a--a---a-$, $a--a--a---a--a-$, $a--a---a--a--a-$, $a---a--a--a--a-$, $-a---a--a--a--a$, $-a--a---a--a--a$, $-a--a--a---a--a$, $-a--a--a--a---a$

Group 3:

$a---a---a--a--a$, $a---a--a---a--a$, $a---a--a--a---a$, $a--a---a---a--a$, $a--a---a--a---a$, $a--a--a---a---a$

Group 4: $a----a--a--a--a$, $a--a----a--a--a$, $a--a--a----a--a$, $a--a--a--a----a$

In Group 1 we must place a b and a c in between each of these as and a b and c at the extremity. Each one can be bc or cb so there are 2^5 different possibilities for each one.

In Group 2 place the b and c between the as with 2 spaces each (3 of them). In the remaining spaces with a 3/1 distribution there are 6 ways of placing either two bs and a c in the group of 3 or two cs and a b (= total $2^3 \cdot 6$).

In Group 3 fill the two space blanks in $2 \cdot 2$ ways. In the remaining two blanks with 3 spaces which can be filled with two bs and a c or the reverse and there are 3 arrangements of each of the two blanks (= total $2^3 \cdot 3^2$).

In Group 4 fill the two blanks with spaces in 2^3 ways, then fill in the four remaining spaces with two as and two bs is $\binom{4}{2} = 6$ ways.

The total number of sequences is then

$$3 \cdot 2^5 + 8 \cdot 2^3 \cdot 6 + 6 \cdot 2^3 \cdot 3^2 + 4 \cdot 2^3 \cdot 6 = 1104.$$

(5) How many bridge hands are there with the suit distribution $6-3-2-2$?

Choose the suit with six cards in 4 ways, choose the suit with three cards in 3 ways, choose six cards from thirteen in first suit in $\binom{13}{6}$ ways, choose three cards in second suit in $\binom{13}{3}$ ways, choose two cards in the third suit in $\binom{13}{2}$ ways twice.

$$4 \cdot 3 \cdot \binom{13}{6}\binom{13}{3}\binom{13}{2}\binom{13}{2} = 35830574208$$

(6) In a bridge deal you and your partner between you have 9 spades. What is the probability that one of your two opponents has exactly three of the four remaining spades?

When I am solving this problem I assume that I know where 26 of the cards are (13 in my hand, 13 in my partner's hand and his is showing face up on the table), this wasn't very clear from the question and you may make other assumptions and solve a different enumeration problem. The remaining 26 cards can be arranged in $\binom{26}{13}$ ways. If we insist that 3 of the remaining 4 spades are in the players hand that is to my left then we choose those three cards from the four spades and then ten cards from the $26 - 4$ cards which are not spades in $\binom{22}{10}$ ways. This is the probability that the hand to my left has 3 spades and 10 non-spades and the probability that that hand has 1 spade and 12 non spades is $\binom{4}{1}\binom{22}{12}$. Therefore the probability that the hand to my left has either 3 spades or 1 spades is

$$\frac{\binom{4}{3}\binom{22}{10} + \binom{4}{1}\binom{22}{12}}{\binom{26}{13}} \simeq .4974$$

Combinatorial proofs: For the following problems give a combinatorial proof by describing a set that is counted by the left hand side of the equality and a set that is counted by the right hand side of the equality and explaining why these two sets are the same.

(1) $\sum_{r=0}^{k} \binom{n}{r}\binom{k}{r} = \binom{n+k}{n}$

Assume that $k \geq n$, if not then prove the equivalent identity $\sum_{r=0}^{n} \binom{n}{r}\binom{k}{r} = \binom{n+k}{k}$.

The right hand side of this equation counts the number of ways of taking n balls from a set of $n + k$ numbered balls labeled 1 through $n + k$ and coloring them red, the remaining k balls are then colored blue.

The left hand side of this equation is graded by r, the the number of balls colored blue that are numbered 1 through n where r is either 0 or 1 or 2 or ... or n. If there are r balls colored blue labeled 1 through n then there must be $k - r$ balls colored blue that were labeled $n + 1$ through $n + k$. There are $\binom{n}{r}$ ways of picking the blue balls from the first n, and $\binom{k}{k-r} = \binom{k}{r}$ ways of choosing the remaining $k - r$ balls to color blue. The remaining n balls will be colored red.

Since both sides of these equations count the number of ways of having k blue balls and n red balls from a set of numbered balls labeled 1 through $n + k$ they count the same thing and hence must be equal.

(2) $\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n} = \binom{2n}{n}$.

We rewrite the left hand side of this equation in summation notation as $\sum_{r=0}^{n} \binom{n-1+r}{r}$. Since $\binom{n-1+r}{r} = \binom{n-1+r}{n-1}$ we can show the equivalent identity

$$\sum_{r=0}^{n} \binom{n-1+r}{n-1} = \binom{2n}{n}$$

The right hand side counts the number of ways of choosing n balls from a set of numbered balls with the labels 1 through $2n$ and coloring those balls red, the remaining n balls will be colored blue.

Now the largest number which appears on one of the red balls is either n or it is labeled with an $n+1$ or it has a label of $n+2$, $\ldots$, or it has label $2n$. The remaining $n-1$ balls are chosen from those labeled with 1 through one less than the maximum label $n+r$ and there are $\binom{n+r-1}{n-1}$ ways of doing this.

Since both sides of this equation count the number of ways at arriving at n red balls and n blue balls out of a total of $2n$ numbered balls, they must be equal.

(3) $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$

Every subset of $r+1$ balls of a set of numbered balls labeled 1 through $n+1$ has maximum label equal to either $r+1$, or $r+2$, or $\ldots$, or $n+1$. If the maximum label of the set is $r+i$ (where i is between 1 and $n-r+1$) then there are $\binom{r+i-1}{r}$ ways of picking the r remaining elements of the set with smaller labels. Therefore the number of ways of taking $n+1$ labeled balls and coloring $r+1$ red will be equal to the sum of $\binom{r+i-1}{r}$ for i from 1 to $n-r+1$ which is the left hand side of this equation. It is equal to the right hand side since they both represent the number of subsets of size $r+1$ of numbered balls labeled 1 through $n+1$.

(4) $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{n-k}$

The left hand side of this equation represents the number of ways of choosing k balls of a set of balls with the labels 1 through n, coloring the $n-k$ balls that were not chosen with the color green, then picking a subset of the k balls just chosen of size ℓ and coloring them red, and coloring the remaining $k-\ell$ balls blue. That is, the left hand side represents the number of partitions of n labeled balls into three sets of $n-k$ green balls, ℓ red balls and $k-\ell$ blue balls.

The right hand side of this equation counts the number of ways of choosing a set of n labeled balls of size ℓ and coloring those balls red, then choosing from the remaining $n-\ell$ balls $n-k$ to color green and then coloring the remaining $k-\ell$ balls and coloring them blue. The result is a partition of n labeled balls into three sets of $n-k$ green balls, ℓ red balls and $k-\ell$ blue balls. Since the outcomes of this procedure are the same as for the left hand side of this equation the two are equal.