HOMEWORK #2 - MATH 4160

DUE: WEDNESDAY FEBRUARY 5, 2002 AT 10:30AM

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups (maximum 3). You need only hand in one assignment per group, and write all names at the top.

Enumeration problems. Do problem #1 and two of #2, #3 or #4:

(1) How many arrangements are there of the letters in the words ELEMENTARY ENUMERATION such that there are no consecutive E's?

Choose where the E's go such that they are not consecutive, to do this instead of choosing from 21 places, 5 to put E's, choose from 17 places, 5 to put E's and then insert a space after the first 4 of those E's This can be done in $\binom{17}{5}$ ways.

Now the number of ways of ordering the remaining letters (2 Ms, 2 Ts, 2 As, 3 Ns, 2 Rs, L, U, Y, I, O) in the remaining 16 spaces is $\frac{16!}{2!2!2!3!2!} = \binom{16}{2}\binom{14}{2}\binom{12}{2}\binom{10}{3}\binom{7}{2}5!$

$$\binom{17}{5}\binom{16}{2}\binom{14}{2}\binom{12}{2}\binom{10}{3}\binom{7}{2}5! = 1348648164864000$$

(2) How many quick pick outcomes are there in the *Super 7* lottery game (pick 7 numbers in 49)? Say 7 winning numbers are fixed and a bonus number is chosen. How many of these quick pick numbers have all 7 numbers correct? 6 numbers and a bonus? 6 numbers and not the bonus? Five numbers?

There are a couple of problems with this question, but if you just ignore them and answer the question as stated and then remark on the errors after the solution.

The number of quick pick outcomes from a game with 49 possible numbers and choose 7 is $\binom{49}{7}$.

If we fix 7 winning numbers and a bonus then there is exactly 1 winning ticket.

If there are 6 numbers right and the bonus number then there are 7 winning tickets.

If there are 6 numbers and the bonus number is wrong then there are $7 \cdot \binom{41}{1} = 287$ winning tickets.

If there are 5 numbers correct then there are $\binom{7}{5}\binom{42}{2} = 18081$ winning tickets.

Remark: Many people asked me if it should be 47 numbers in the Super 7 lottery. I don't care. If you answered it with these numbers then subtract 2 from every number larger than 40 in the answers above.

(3) The following questions are regarding a poker hand of five cards.

(a) How many hands contain a pair and three more cards that have the same suit as one of the cards in the pair? (e.g. $3\heartsuit, 3\clubsuit, 6\clubsuit, 8\clubsuit, Q\clubsuit$)

Choose the pair value $\binom{13}{1}$, choose the suits that appear $\binom{4}{2}$, choose the suit for the other three cards $\binom{2}{1}$, choose the values of the other three cards $\binom{12}{3}$.

$$\binom{13}{1}\binom{4}{2}\binom{2}{1}\binom{12}{3} = 34320$$

(b) How many hands contain a pair and three more cards in sequence of the same suit (which may or may not be the same as the suit in the pair)? (e.g. $5\heartsuit, 5\spadesuit, J\clubsuit, Q\clubsuit, K\clubsuit$ or $7\heartsuit, 7\diamondsuit, 2\diamondsuit, 3\diamondsuit, 4\diamondsuit$) The sequence of 3 should not share a card with the pair or even have a card in common with the pair (e.g. the hand $5\heartsuit, 6\heartsuit, 7\heartsuit, 6\clubsuit, 6\diamondsuit$ is not valid).

For this problem pick the suits first and the values of the cards second (because the suits are easy to do and the other part is less trivial). For the pair, pick two suits to appear in $\binom{4}{2}$ ways and pick the suit for the sequence in $\binom{4}{1}$ ways. Now one can almost pick the pair and the sequence independently, but the problem is if one picks a pair of A,2 or K there are 10 possible sequences, for a pair of $3, 4, 5, 6, 7, 8, 9, 10, J, Q$ there are 9 possible sequences. In total there are

$$\binom{4}{2}\binom{4}{1}(3 \cdot 10 + 10 \cdot 9) = 2880$$

valid hands of this type.
Another way of solving this problem (and probably better) is to pick the sequence of three cards first and then choose the pair. There are $\binom{4}{1}$ suits for the sequence and $\binom{12}{1}$ different sequences then there are $\binom{10}{1}$ possible values for the pair and $\binom{4}{2}$ possible ways of choosing two suits which determine the pair.

$$\binom{4}{1}\binom{12}{1}\binom{10}{1}\binom{4}{2} = 2880$$

(4) There are 100 pennies, how many ways are there of distributing them to 5 different people (i.e. each person may or may not get some of these pennies)? How many ways are there of distributing these pennies so that each person gets at least 5 pennies each? (Hint: think of the derivative problem from last time).

Distribute the pennies in $\binom{100+5-1}{5-1} = 4598126$ ways. For the explanation see the last homework.

To make sure that everyone gets at least 5 pennies, give everyone 5 pennies first and then distribute the remaining 75 pennies in $\binom{75+5-1}{5-1} = 1502501$ ways.

Do three of the following 6 problems. These will use the principle of inclusion-exclusion.

(1) How many n-digit decimal sequences (using the digits $0-9$) are there in which the digits $1, 2$ and 3 all appear?

Let A_i = set of sequences of n digits where i does not appear. The number of n digit decimal sequences = the total number of decimal sequences minus those that do not have either 1, 2 or 3. That is we wish to calculate $10^n - |A_1 \cup A_2 \cup A_3| = 10^n - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$ where 10^n represents the number of sequences of n digits and $|A_1 \cup A_2 \cup A_3|$ represents the number of n digit sequences that either do not have a 1 or a 2 or a 3. $|A_i| = 9^n$, $|A_i \cap A_j| = 8^n$ and $|A_1 \cap A_2 \cap A_3| = 7^n$. The answer is then

$$10^n - 3 \cdot 9^n + 3 \cdot 8^n - 7^n$$

(2) How many ways are there of rolling a sided die 10 times in a sequence such that all 6 faces appear at least once?

Let A_i = set of sequences of 10 die rolls where i does not appear. We wish to count 6^{10} minus the number of sequences of 10 rolls where at least one of the 6 faces does not appear $= 6^{10} - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6|$. $|A_i| = 5^{10}$, $|A_i \cap A_j| = 4^{10}$, $|A_i \cap A_j \cap A_k| = 3^{10}$, $|A_i \cap A_j \cap A_k \cap A_\ell| = 2^{10}$, $|A_i \cap A_j \cap A_k \cap A_\ell \cap A_n| = 1^{10}$ and the intersection of all six sets is 0. Therefore the number of sequences where all 6 faces appear at least once is

$$= 6^{10} - 6 \cdot 5^{10} + \binom{6}{2} \cdot 4^{10} - \binom{6}{3} \cdot 3^{10} + \binom{6}{4} \cdot 2^{10} - \binom{6}{5} 1^{10} = 16435440$$

(3) How many positive integers ≤ 420 are relatively prime to $420 = 5 \cdot 2^2 \cdot 7 \cdot 3$?

Let A_k be the set of positive integers less than or equal to 420 which are divisible by k. The positive integers which are relatively prime to 420 and less than or equal to are those which are not divisible by either 2 or 5 or 7 or 3. That is we wish to calculate $420 - |A_2 \cup A_3 \cup A_5 \cup A_7| = 420 - |A_2| - |A_3| - |A_5| - |A_7| + |A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7| - |A_2 \cap A_3 \cap A_5| - |A_2 \cap A_3 \cap A_7| - |A_2 \cap A_5 \cap A_7| - |A_3 \cap A_5 \cap A_7| + |A_2 \cap A_3 \cap A_5 \cap A_7|$. $|A_2| = 420/2$, $|A_2 \cap A_3| = \frac{420}{2 \cdot 3}$, ect. and for all of these sets

$$\left| \bigcap_{i=1}^{r} A_{p_i} \right| = \frac{420}{\prod_{i=1}^{r} p_i}$$

where $p_i \in \{2, 3, 5, 7\}$. Therefore the number of positive integers less than or equal to 420 which are relatively prime to 420 is

$$420 - 210 - 140 - 84 - 60 + 70 + 42 + 30 + 28 + 20 + 12 - 14 - 10 - 6 - 4 + 2 = 96$$

(4) How many arrangements of 52 letters, 2 As, 2 Bs, 2 Cs, etc. with no pair of consecutive letters the same?

Let A_X be the set of arrangements of 52 letters with two of each letter where the Xs appear next to each other (X may represent any letter of the alphabet). The number of arrangements of 52 letters with no pair of consecutive letters the same = the number of 52 arrangements - the number of arrangements with at least one pair of letters adjacent $= 52!/2^{26} - |A_A \cup A_B \cup A_C \cup \cdots A_Y \cup A_Z|$. Now the intersection of k different subsets A_{L_i} is counted by

$$|A_{L_1} \cap A_{L_2} \cap \cdots \cap A_{L_k}| = \binom{52-k}{k} k! \frac{(52-2k)!}{2^{26-k}}$$

(choose k spaces to put the pairs of letters, order the pairs, and then order the remaining $52-2k$ letters). Therefore the total number of arrangements of the 52 letters in this way is

$$52!/2^{26} - \sum_{k=1}^{26} (-1)^{k-1} \binom{26}{k} \binom{52-k}{k} k! \frac{(52-2k)!}{2^{26-k}}$$
$$= 43784166376626941676867747083192465648028709600296960000000$$
$$\approx 4.378 \times 10^{59}$$

(5) How many ways are there of dealing a 13 card hand with at least suit that does not appear?

Let A_* be the set of thirteen card hands with no $*$ where $* \in \{\heartsuit, \spadesuit, \diamondsuit, \clubsuit\}$. We are counting $|A_\heartsuit \cup A_\spadesuit \cup A_\diamondsuit \cup A_\clubsuit|$ and we know that $|A_*| = \binom{39}{13}$ and $|A_* \cap A_\#| = \binom{26}{13}$ and $|A_* \cap A_\# \cap A_\&| = 1$ and the intersection of all four sets is empty. Therefore the number of 13 card hands with a void in at least one suit is

$$4\binom{39}{13} - 6\binom{26}{13} + 4 = 32427298180$$

(6) How many 13 card hands have at least one type of face card (J, Q, K, A)?

The number of ways of having a 13 card hand with at least one face card = the number of 13 card hands minus the number of hands which contain no face cards (since there are 36 non-face cards there are $\binom{36}{13}$ non-face card hands). This is best solved without using the inclusion-exclusion principle and we should say then that it is equal to

$$\binom{52}{13} - \binom{36}{13} = 632702770000$$