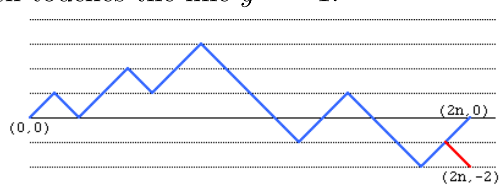
HOMEWORK #3 SOLUTIONS - MATH 4160

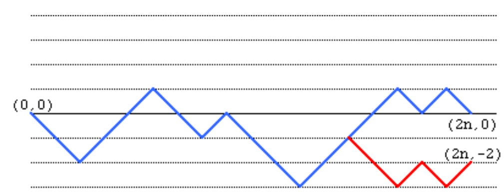
(1) How many paths are there that start at the point $(0,0)$ and go to the point $(2n,0)$ by taking steps in a NE or SE direction (i.e. $(+1,+1)$ or $(+1,-1)$)?

Solution: Every path that starts at $(0,0)$ and ends at $(2n,0)$ has the same number of NE steps and SE steps (because the y coordinate is 0). Every path of this type is a sequence of $2n$ steps, n of which are NE and n of which are SE, so it suffices to choose which of the $2n$ steps are NE and the rest will be SE and this determines the path. Therefore there are $\binom{2n}{n}$ paths of this type.

(2) Bonus: How many of these paths go below the line $y=0$?

Solution: Every path which passes below the line $y=0$ touches the line $y=-1$ at least once so we are counting the number of paths that touch the line $y=-1$. Reflect the path in the line $y=-1$ from the last time that it touches this line to the end of the path. Two examples are seen below where the red path is the part of the path after the last time the path touches the line $y=-1$.





Notice that the reflected path starts at $(0,0)$ and ends at $(2n,-2)$ and there are the same number of paths which start at $(0,0)$ and end at $(2n,-2)$ as there are paths which start at $(0,0)$ and end at $(2n,0)$ and touch the line $y=-1$ at least once. Therefore we are counting the number of paths which start at $(0,0)$ and end at $(2n,-2)$ taking NE and SE steps only. We know that these path must take 2 more SE steps than NE steps so the there are $n-1$ NE steps and $n+1$ SE steps and there are $\binom{2n}{n-1}$ such paths.

(3) Let $p_{\leq k}(n)$ represent the number of partitions of n with width less than or equal to k. Let $P_{\leq k}(x) = \sum_{n\geq 0} p_{\leq k}(n)x^n$ be the generating function for this sequence. Prove by induction on k that generating function for this sequence is

$$P_{\leq k}(x) = \prod_{r=1}^{k} \frac{1}{1-x^r}.$$

Solution: Every partition of width less than or equal to k either has no parts of size k (there are $p_{\leq k-1}(n)$ such partitions with this property) or it is a partition of size $n-k$ and width less than or equal to k sitting on a part of size k (there are $p_{\leq k}(n-k)$ such partitions). Since these two outcomes are disjoint we have that

$$p_{\leq k}(n) = p_{\leq k-1}(n) + p_{\leq k}(n-k).$$

Let $P_{\leq k}(x) = \sum_{n\geq 0} p_{\leq k}(n)x^n$ be a generating function for the number of partitions with width less than or equal to k. Since there is exactly one partition of n with width ≤ 1 for every n (namely the partition with n parts of size 1), we have that

$$P_{\leq 1}(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

Now we also have that

$$\begin{aligned} P_{\leq k}(x) &= \sum_{n\geq 0} p_{\leq k}(n)x^n = \sum_{n\geq 0} (p_{\leq k-1}(n) + p_{\leq k}(n-k))x^n \\ &= \sum_{n\geq 0} p_{\leq k-1}(n) + \sum_{n\geq 0} p_{\leq k}(n-k)x^n = P_{\leq k-1}(x) + x^k P_{\leq k}(x) \end{aligned}$$

Thus the recursion on the coefficients is equivalent to the statement $P_{\leq k}(x) - x^k P_{\leq k}(x) = P_{\leq k-1}(x)$. This means that

$$P_{\leq k}(x)(1-x^k) = P_{\leq k-1}(x)$$

and therefore

$$P_{\leq k}(x) = P_{\leq k-1}(x)\frac{1}{1-x^k}.$$

Now assume by induction that the generating function $P_{\leq k-1}(x) = \prod_{r=1}^{k-1} \frac{1}{1-x^r}$, then

$$P_{\leq k}(x) = \frac{1}{1-x^k} P_{\leq k-1}(x) = \frac{1}{1-x^k} \prod_{r=1}^{k-1} \frac{1}{1-x^r} = \prod_{r=1}^{k} \frac{1}{1-x^r}.$$

Since have shown that $P_{\leq k-1}(x) = \prod_{r=1}^{k-1} \frac{1}{1-x^r}$ implies that $P_{\leq k}(x) = \prod_{r=1}^{k} \frac{1}{1-x^r}$ and we know for the base case that $P_{\leq 1}(x) = \frac{1}{1-x}$, then it follows by induction

$$P_{\leq k}(x) = \prod_{r=1}^{k} \frac{1}{1-x^r}$$

for all $k > 0$.

(4) Define the Fibonacci numbers as the sequence satisfying $f_0 = 1$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Find an algebraic expression for the generating function $F(x) = \sum_{n\geq 0} f_n x^n$.

Solution:

$$\begin{aligned} F(x) &= \sum_{n\geq 0} f_n x^n = f_0 + f_1 x + \sum_{n\geq 2} f_n x^n \\ &= 1 + x + \sum_{n\geq 2} (f_{n-1} + f_{n-2})x^n \\ &= 1 + x + \sum_{n\geq 2} f_{n-1}x^n + \sum_{n\geq 2} f_{n-2}x^n \\ &= 1 + xF(x) + x^2 F(x) \end{aligned}$$

This implies that $F(x) - xF(x) - x^2F(x) = 1$ and so $F(x) = \frac{1}{1-x-x^2}$.