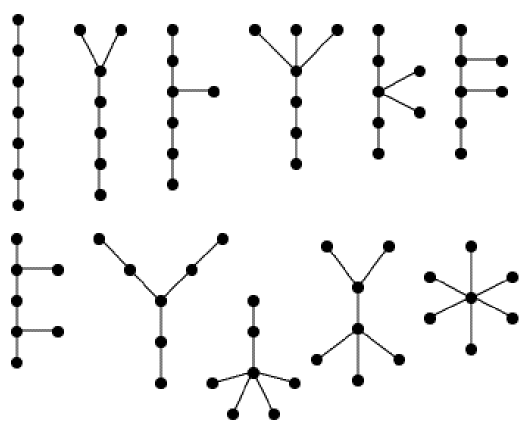
HOMEWORK #4 - MATH 4160

DUE: FRIDAY MARCH 7, 2002 AT 10:30AM

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups (maximum 3). You need only hand in one assignment per group, and write all names at the top.

Enumeration problems.

(1) How many different ways are there of labeling the vertices of the following trees so that the graphs are not isomorphic? Hint: A theorem due to Cayley says the total number of labeled trees on n vertices is n^{n-2} so if you correctly identify the number of labeled trees of each of the 11 types and add the values together you should get that there are $7^5 = 16807$ in total.



(2) How many paths are there from the point $(0, n)$ to the point $(k, 0)$ using only SOUTH, EAST and SOUTHEAST steps (that is, $(0, -1)$, $(+1, 0)$ and $(+1, -1)$ steps)? How many of these paths are there of this type if there are exactly ℓ south-east steps and $n - \ell$ south steps and $k - \ell$ east steps?
(3) How many 7 card hands from a 52 card deck have either a three-of-a-kind or a four-of-a-kind or both?
(4) Say that there are 100 balls in a bin with 40 colored green and 60 colored blue. Reach in and pull out 10 of these balls at random. What is the probability that either 0, 1 or 2 of the balls that are selected are blue and the others are green?

Generating functions:

(1) Say that $A(x)$ is the generating function for the sequence $(a_0, a_1, a_2, \ldots)$ so that $A(x) = \sum_{n\geq 0} a_n x^n$ and that $B(x) = \sum_{n\geq 0} b_n x^n$ is the generating function for the sequence $(b_0, b_1, b_2, \ldots)$ where the numbers a_i represent the number of *widgets* of size i and b_j represents the number of *doodles* of size j. What does the quantity $a_i b_j$ represent (a combinatorial interpretation)? Compute the coefficient of x^n in $A(x)B(x)$ and $A(x) + B(x)$ and give a combinatorial interpretation of this coefficient.
(2) Give a recurrence for the number of sequences of 0s and 1s where every 1 is followed by an odd number of 0s. Use this to derive a generating function for the number of sequences of 0s and 1s where every 1 is followed by an odd number of 0s.
(3) Consider the paths that take steps either NORTHEAST $(+1, +1)$ or SOUTHEAST $(+1, -1)$ that start at $(0, 0)$ and after $2n$ steps end at $(2n, 0)$. The paths of this sort which do not go below the line $y = 0$ are called Dyck paths. Let C_n represent the number of Dyck paths of length n. Give a combinatorial proof that the numbers C_n must satisfy the recurrence

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

with $C_0 = C_1 = 1$. Calculate C_2 through C_6. Use this recurrence to derive a generating function for the numbers C_n.

Bonus (graduate students should do these):

(1) For a sequence $(a_0, a_1, a_2, \ldots)$ the exponential generating function is defined as the power series $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$. The exponential generating function also encodes information about a sequence and it can also be used to derive formulas in the same way that we have with generating functions. Let $A(x)$ represent the exponential generating function for the sequence of $(a_0, a_1, a_2, \ldots)$ where a_i represents the number of *widgets* of size i and $B(x)$ represents the exponential generating function for the sequence $(b_0, b_1, b_2, \ldots)$ where b_j is the number of *doodles* of size j. Compute the coefficient of $\frac{x^n}{n!}$ in the expression $A(x)B(x)$ and $A(x) + B(x)$ and give a combinatorial interpretation for these coefficients.
(2) Let $a_0 = 0$, $a_1 = 1$ and $a_n = 3a_{n-1} - a_{n-2} + 2$. Using generating functions derive at least one non-recursive formula for a_n as we did in class for the Fibonacci numbers.