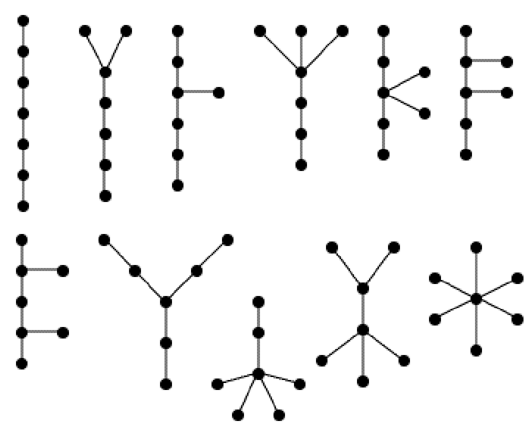
HOMEWORK #4 SOLUTIONS - MATH 4160

DUE: FRIDAY MARCH 7, 2002 AT 10:30AM

Enumeration problems.

(1) How many different ways are there of labeling the vertices of the following trees so that the graphs are not isomorphic? Hint: A theorem due to Cayley says the total number of labeled trees on n vertices is n^{n-2} so if you correctly identify the number of labeled trees of each of the 11 types and add the values together you should get that there are $7^5 = 16807$ in total.



Solution: Number the graphs from left to right in the first row, then left to right in the second row. Below I will explain how to 'count' the number of non-isomorphic labelings of the graph.

1^{st} tree: order the vertices from top to bottom but since the graph is the same when it is turned upside down $= 7!/2 = 2520$

2^{nd} tree: pick the vertex of degree 3 (7 ways), pick the two vertices of degree 1 ($\binom{6}{2}$ ways), order the remaining 4 vertices $= 7 \cdot \binom{6}{2} \cdot 4! = 2520$.

3^{rd} tree: Pick the vertex of degree 3 (7 ways), pick the vertex of degree 1 (6 ways), on the branch of length 2 from this vertex pick the first and second vertex ($5 \cdot 4$ ways), on the branch of length three order the remaining 3 vertices (3! ways) $= 7! = 5040$.

4^{th} tree: Pick the vertex of degree 4 (7 ways), pick the three vertices of degree 1 ($\binom{6}{3}$ ways), order the remaining labels of the branch of length $3 = 7 \cdot \binom{6}{3} \cdot 3! = 840$.

5^{th} tree: Pick the center vertex (7 ways), choose two labels for the vertices of degree 1 ($\binom{6}{2}$ ways), choose two labels that go next two the vertex of degree 4 ($\binom{4}{2}$ ways), choose a vertex which adjoins the larger of the two labels (2 ways), the remaining label goes on the remaining vertex $= 7\binom{6}{2}\binom{4}{2}2 = 1260$.

6^{th} tree: Pick the vertex of degree 3 with 2 leaves (7 ways), choose two labels for the two leaves ($\binom{6}{2}$ ways), choose and order the vertices along the path of length 3 ($\binom{4}{3}3!$ ways), the last label goes on the vertex of degree 1 which is unlabled $= 7\binom{6}{2}4! = 2520$.

7^{th} tree: Pick the center vertex (7 ways), pick 2 labels two go on the vertices that are next to the center ($\binom{6}{2}$ ways), pick two labels that will be attached to the larger of these two labels ($\binom{4}{2}$ ways), the remaining labels are on the last two vertices $= 7\binom{6}{2}\binom{4}{2} = 630$.

8^{th} tree: Pick the center vertex (7 ways), choose 3 labels for the vertices of degree 2 ($\binom{6}{3}$ ways), choose one label to go on the leaf next to the highest label (3 ways), choose one label to go on the leaf next to the second highest label (2 ways), the last label is attached to the last leaf $= 7\binom{6}{3} \cdot 3 \cdot 2 = 840$.

9^{th} tree: Pick the vertex of degree 5 (7 ways), choose 4 labels to go on the leaves ($\binom{6}{4}$ ways), order the two remaining labels on the path of length 2 (2 ways) $= 7\binom{6}{4}2 = 210$.

10^{th} tree: Pick the vertex of degree 4 (7 ways), choose 3 labels for the leaves next to this label ($\binom{6}{3}$ ways), choose one label for the vertex of degree 3 (3 ways), the remaining labels are on the last two leaves $= 7\binom{6}{3}3 = 420$

11^{th} tree: Pick the center vertex (7 ways), the remaining labels go on the 6 leaves = 7.

Note: we check that $2520+2520+5040+840+1260+2520+630+840+210+420+7 = 16807 = 7^5$. This agrees with Cayley's theorem.

(2) How many paths are there from the point $(0, n)$ to the point $(k, 0)$ using only SOUTH, EAST and SOUTHEAST steps (that is, $(0, -1)$, $(+1, 0)$ and $(+1, -1)$ steps)? How many of these paths are there of this type if there are exactly ℓ south-east steps and $n - \ell$ south steps and $k - \ell$ east steps?

If there are exactly ℓ south-east steps, $n - \ell$ south steps and $k - \ell$ east steps then there are $\binom{n+k-\ell}{\ell}\binom{n+k-2\ell}{n-\ell}\binom{k-\ell}{k-\ell}$ possible paths. For any path, there can be at most $min(n, k)$ SE steps and there can be a few as 0 SE steps. Therefore the total number of paths of this type that start at $(0, n)$ and end at $(k, 0)$ is

$$\sum_{\ell=0}^{min(n,k)} \binom{n+k-\ell}{\ell}\binom{n+k-2\ell}{n-\ell}.$$

I don't know if this expression can be simplified any (probably it cannot).

(3) How many 7 card hands from a 52 card deck have either a three-of-a-kind or a four-of-a-kind or both?

Solution: This can be done with inclusion-exclusion, but we must be very careful. Let A = the set of 7 card hands with a 3-of-a-kind and B = the set of 7 card hands with a 4-of-a-kind. We wish to find $|A \cup B| = |A| + |B| - |A \cap B|$. The difficulty with counting the number of hands with a 3-of-a-kind is that if we just pick the 3-of-a-kind and 4 other cards then we have over counted since the hands with two 3-of-a-kinds were counted twice. Therefore, $|A|$ = (pick the card for the three-of-a-kind) (pick the 3 suits of 4 that appear in the three-of-a-kind) (pick 4 other cards) - (pick two types of cards for the three-of-a-kind) (pick 3 suits to appear in the first one) (pick 3 suits to appear in the second one) (pick one remaining card) = $\binom{13}{1}\binom{4}{3}\binom{48}{4} - \binom{13}{2}\binom{4}{3}^2\binom{44}{1} = 10118160 - 54912 = 10063248$. $|B|$ = (pick the card for the four-of-a-kind) (choose 3 other cards) = $\binom{13}{1}\binom{48}{3} = 224848$. $|A \cap B| = \binom{13}{1}\binom{4}{3}\binom{12}{1} = 624$. Therefore

$$|A \cup B| = 10063248 + 224848 - 624 = 10287472$$

There is a second way of doing this problem. The hands that contain either a 3-of-a-kind or a 4-of-a-kind or both are those that are not of the form 3 pairs and a singleton, 2 pairs and 3 singletons, 1 pair and 5 singletons, or 7 seven singletons. That means that that we should subtract the number of these hands from the total number of hands.

$$\binom{52}{7} - \left(\binom{13}{3}\binom{10}{1}\binom{4}{2}^3\binom{4}{1} + \binom{13}{2}\binom{11}{3}\binom{4}{2}^2\binom{4}{1}^3 + \binom{13}{1}\binom{12}{5}\binom{4}{2}\binom{4}{1}^5 + \binom{13}{7}\binom{4}{1}^7\right)$$
$$= 10287472$$

No matter how you count these hands you should get the same number.

(4) Say that there are 100 balls in a bin with 40 colored green and 60 colored blue. Reach in and pull out 10 of these balls at random. What is the probability that either 0, 1 or 2 of the balls that are selected are blue and the others are green?

Solution: The easiest way of thinking about this problem is to assume that the 100 balls are numbered and the first 40 are green while the last 60 are colored blue. There are $\binom{100}{10}$ ways of picking 10 balls from the bin. There are $\binom{60}{0}\binom{40}{10} + \binom{60}{1}\binom{40}{9} + \binom{60}{2}\binom{40}{8}$ ways of choose these balls such that either $0, 1$ or 2 are blue. The probability is then

$$\frac{\binom{60}{0}\binom{40}{10} + \binom{60}{1}\binom{40}{9} + \binom{60}{2}\binom{40}{8}}{\binom{100}{10}} = \frac{153375285778}{17310309456440} \approx .008860343379$$

Generating functions:

(1) Say that $A(x)$ is the generating function for the sequence $(a_0, a_1, a_2, \ldots)$ so that $A(x) = \sum_{n\geq 0} a_n x^n$ and that $B(x) = \sum_{n\geq 0} b_n x^n$ is the generating function for the sequence $(b_0, b_1, b_2, \ldots)$ where the numbers a_i represent the number of *widgets* of size i and b_j represents the number of *doodles* of size j. What does the quantity $a_i b_j$ represent (a combinatorial interpretation)? Compute the coefficient of x^n in $A(x)B(x)$ and $A(x) + B(x)$ and give a combinatorial interpretation of this coefficient.

Solution: By the multiplication principle $a_i b_j$ is equal to the number of pairs (x, y) where x is a widget of size i and y is a doodle of size j.

The coefficient of x^n in $A(x)B(x)$ is $\sum_{k=0}^{n} a_k b_{n-k}$ and by the addition principle this represents the number of pairs (x, y) where x is a widget of size k and y is a doodle of size $n - k$ where k is some number between 0 and n. In other words this quantity represents the number of pairs of integers (x, y) where x is a doodle and y is a widget and the size of x plus the size of y is n.

The coefficient of x^n in $A(x) + B(x)$ is $a_n + b_n$ and this represents the disjoint union of the set of widgets of size n and the doodles of size n.

(2) Give a recurrence for the number of sequences of 0s and 1s where every 1 is followed by an odd number of 0s. Use this to derive a generating function for the number of sequences of 0s and 1s where every 1 is followed by an odd number of 0s.

Solution: There are at least two recurrences that you can give for this problem. One would come from noting that every sequence of odd length begins with either a 00 or a 01 followed by a sequence of length $n-2$ with every 1 followed by an odd number of 0s (so if n odd then $a_n = 2a_{n-2}$) and every sequence of even length either begins with a 00 or a 10 followed by a sequence of length $n-2$ with every 1 followed by an odd number of 0s (so if n is even then $a_n = 2a_{n-2}$). Therefore

$$a_n = 2a_{n-2}$$

for all $n > 1$ and $a_0 = 1$ and $a_1 = 1$. From this we arrive at

$$\begin{aligned} A(x) &= \sum_{n\geq 0} a_n x^n = 1 + x + \sum_{n\geq 2} a_n x^n \\ &= 1 + x + \sum_{n\geq 2} (2a_{n-2}) x^n \\ &= 1 + x + 2x^2 A(x) \end{aligned}$$

Therefore $A(x) - 2x^2 A(x) = 1 + x$ and thus $A(x) = \frac{1+x}{1-2x^2}$.

The second solution comes from noting that every sequence of this type is either a sequence of n 0s or it is a sequence of this type of length $n-2$ followed by a 10 or a sequence of this type of length $n-4$ followed by 1000 or a sequence of this type of length $n-6$ followed by 1 and five 0s, etc. In other words, we have

$$a_n = 1 + a_{n-2} + a_{n-4} + a_{n-6} + \cdots$$

and we again have $a_0 = 1$ and $a_1 = 1$. This means that

$$\begin{aligned} A(x) &= \sum_{n\geq 0} a_n x^n = 1 + x + \sum_{n\geq 2} a_n x^n \\ &= 1 + x + \sum_{n\geq 2} (1 + a_{n-2} + a_{n-4} + a_{n-6} + \cdots) x^n \\ &= 1 + x + \sum_{n\geq 2} x^n + \sum_{n\geq 2} \sum_{k\geq 1} a_{n-2k} x^n \\ &= \frac{1}{1-x} + \sum_{k\geq 1} x^{2k} A(x) \\ &= \frac{1}{1-x} + \frac{x^2}{1-x^2} A(x). \end{aligned}$$

Now if we solve this equation for $A(x)$ we see that

$$A(x) = \frac{\frac{1}{1-x}}{1 - \frac{x^2}{1-x^2}} = \frac{\frac{1}{1-x}}{\frac{1-2x^2}{1-x^2}} = \frac{1+x}{1-2x^2}.$$

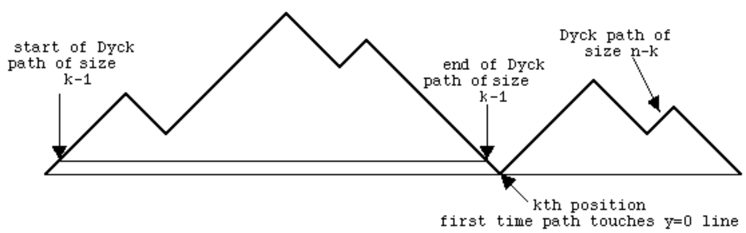
And this is the same answer we had before.

(3) Consider the paths that take steps either NORTHEAST $(+1, +1)$ or SOUTHEAST $(+1, -1)$ that start at $(0, 0)$ and after $2n$ steps end at $(2n, 0)$. The paths of this sort which do not go below the line $y = 0$ are called Dyck paths. Let C_n represent the number of Dyck paths of length n. Give a combinatorial proof that the numbers C_n must satisfy the recurrence

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

with $C_0 = C_1 = 1$. Calculate C_2 through C_6. Use this recurrence to derive a generating function for the numbers C_n.

Solution: The number of Dyck paths which do not touch the line $y = 0$ until the k^{th} down step is $C_{k-1}C_{n-k}$ because if we look at the picture for one of these paths we see that the first step must be an up step, then there is a Dyck path of size $k-1$ then a down step to touch the $y = 0$ line and then a Dyck path of size $n - k$.



Since every Dyck path first touches the $y = 0$ line either at $k = 1$ or $k = 2$ or $\cdots$ or $k = n$ we have that

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}.$$

C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
1	1	2	5	14	42	132	429	1430

These are the Catalan numbers. Let $C(x) = \sum_{n\geq 0} C_n x^n$ be a generating function for these numbers. Since the coefficient of x^n in $C(x)C(x)$ is $\sum_{k=0}^{n} C_k C_{n-k}$ (see the first problem in this section) then the coefficient of x^n in $xC(x)C(x)$ is exactly the right hand side of this equation. Therefore

$$C(x) = 1 + xC(x)C(x)$$

This means that $xC(x)^2 - C(x) + 1 = 0$.

Use the quadratic formula $\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a} \text{ with } a = x, b = -1 \text{ and } c = 1\right)$ to come up with two possible solutions for $C(x)$.

$$C(x) = \frac{1+\sqrt{1-4x}}{2x} \text{ or } \frac{1-\sqrt{1-4x}}{2x}$$

Only one of these is the real generating function. To determine which one we need to find the first few terms of the taylor expansion of these expressions. Note that $\sqrt{1-4x} = 1 - 2x +$ other terms, so

$$\frac{1-\sqrt{1-4x}}{2x} = (1 - (1 - 2x + \text{ other terms}))/(2x) = 1 + \text{ other terms}$$

while

$$\frac{1+\sqrt{1-4x}}{2x} = (1 + (1 - 2x + \text{ other terms}))/(2x) = \frac{1}{x} - 1 + \text{ other terms}$$

so only the first one of these is correct. Therefore $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Bonus (graduate students should do these):

(1) For a sequence $(a_0, a_1, a_2, \ldots)$ the exponential generating function is defined as the power series $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$. The exponential generating function also encodes information about a sequence and it can also be used to derive formulas in the same way that we have with generating functions. Let $A(x)$ represent the exponential generating function for the sequence of $(a_0, a_1, a_2, \ldots)$ where a_i represents the number of *widgets* of size i and $B(x)$ represents the exponential generating function for the sequence $(b_0, b_1, b_2, \ldots)$ where b_j is the number of *doodles* of size j. Compute the coefficient of $\frac{x^n}{n!}$ in the expression $A(x)B(x)$ and $A(x) + B(x)$ and give a combinatorial interpretation for these coefficients.

Solution: The coefficient of $\frac{x^n}{n!}$ in $A(x) + B(x)$ is simply $a_n + b_n$ which we have already established is the number of objects which are either widgets or doodles of size n (or the disjoint union of the set of widgets and doodles).

The coefficient of $\frac{x^n}{n!}$ in $A(x)B(x)$ is the quantity $\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ which is the number of ways of starting with a set of n labeled objects and picking k of them (where k ranges between 0 and n) and turning that subset into a widget of size k and taking the remaining $n-k$ objects are turning those elements into a doodle of size $n-k$. There are other similar interpretations for this expression, but this one is the basis for the 'theory of Species.'

Another possible answer is it this coefficient that it counts triples (S, A, B) where S is a subset of $\{1, 2, \ldots, n\}$, A is a widget size $|S|$ and B is a doodle of size $n - |S|$.

(2) Let $a_0 = 0$, $a_1 = 1$ and $a_n = 3a_{n-1} - a_{n-2} + 2$. Using generating functions derive at least one non-recursive formula for a_n as we did in class for the Fibonacci numbers.

This problem is completely straightforward, but it requires a huge computation.

$$\begin{aligned}
A(x) &= \sum_{n\geq 0} a_n x^n = 0 + x + \sum_{n\geq 2} a_n x^n \\
&= x + \sum_{n\geq 2}(3a_{n-1} - a_{n-2} + 2)x^n \\
&= x + \sum_{n\geq 2} 2x^n + \sum_{n\geq 2} 3a_{n-1}x^n - \sum_{n\geq 2} a_{n-2}x^n \\
&= x + 2\frac{x^2}{1-x} + 3x(A(x) - a_0) - x^2 A(x) \\
&= \frac{x+x^2}{1-x} + (3x - x^2)A(x)
\end{aligned}$$

Therefore $A(x)$ must have the expression

$$A(x) = \frac{x+x^2}{(1-x)(1-3x+x^2)}.$$

Now we will try to manipulate this and get an equivalent expression. I will use partial fraction decomposition but I will do it in two steps because I want to find two expressions for the coefficients a_n.

$$x\frac{1+x}{(1-x)(1-3x+x^2)} = x\frac{A}{1-x} + x\frac{Bx+C}{1-3x+x^2}$$

If we solve for A, B and C we find that

$$\begin{aligned}
\frac{x+x^2}{(1-x)(1-3x+x^2)} &= \frac{-2x}{1-x} + \frac{3x-2x^2}{1-3x+x^2} \\
&= -2\sum_{n\geq 1} x^n + \frac{3x-2x^2}{1-(3x-x^2)} \\
&= -2\sum_{n\geq 1} x^n + (3x-2x^2)\sum_{n\geq 0} x^n(3-x)^n \\
&= -2\sum_{n\geq 1} x^n + (3x-2x^2)\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n-k}x^{n+k}
\end{aligned}$$

For $m > 1$ the coefficient of x^m in the first term is -2. For $m > 1$ the coefficient of x^m in the expression $3x\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n-k}x^{n+k}$ is

$$3\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor-1}(-1)^k\binom{m-k-1}{k}3^{m-2k-1}$$

For $m > 2$, the coefficient of x^m in the expression $-2x^2\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n-k}x^{n+k}$ is

$$-2\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor-1}(-1)^k\binom{m-k-2}{k}3^{m-2k-2}$$

Therefore we have one expression for a_n given as

$$a_n = -2 + 3 \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^k \binom{m-k-1}{k} 3^{m-2k-1} - 2 \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^k \binom{m-k-2}{k} 3^{m-2k-2}$$

OK. So it is a little complicated, but it is something I could use to compute a value by hand fairly easily. For example, if $m = 6$ we have

$$a_6 = -2 + 3\left(\binom{5}{0}3^5 - \binom{4}{1}3^3 + \binom{3}{2}3^1\right) - 2\left(\binom{4}{0}3^4 - \binom{3}{1}3^2 + \binom{2}{2}\right) = 320$$

This is not the only formula that I can arrive at. Notice that $1 - 3x + x^2$ factors to $1 - 3x + x^2 = (1 - \phi x)(1 - \overline{\phi} x)$ where $\phi = \frac{3+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{3-\sqrt{5}}{2}$. In fact $\phi\overline{\phi} = 1$ and $\phi + \overline{\phi} = 3$. Using partial fraction decomposition we see that for unknown A and B satisfying

$$\frac{3 - 2x}{(1 - \phi x)(1 - \overline{\phi} x)} = \frac{A}{1 - \phi x} + \frac{B}{1 - \overline{\phi} x},$$

then $3 - 2x = A(1 - \overline{\phi} x) + B(1 - \phi x)$ so $A + B = 3$ and $A\overline{\phi} + B\phi = 2$. It works out very nicely that $A = \phi$ and $B = \overline{\phi}$. Therefore

$$A(x) = \frac{-2x}{1-x} + \frac{3x - 2x^2}{1 - 3x + x^2} = \frac{-2x}{1-x} + \frac{\phi x}{1 - \phi x} + \frac{\overline{\phi} x}{1 - \overline{\phi} x}$$
$$= -2 \sum_{n \geq 1} x^n + \sum_{n \geq 1} \phi^n x^n + \sum_{n \geq 1} \overline{\phi}^n x^n.$$

Now take the coefficient of x^m for $m \geq 1$ in this expression and we see that

$$a_m = \phi^m + \overline{\phi}^m - 2.$$