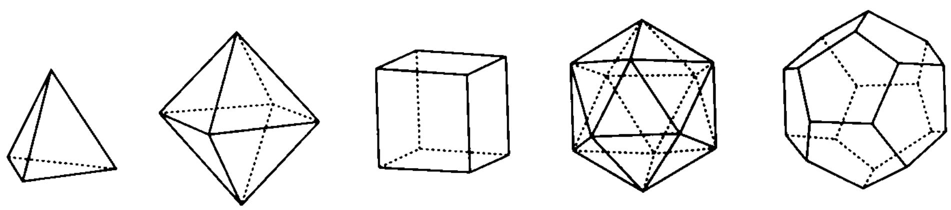
HOMEWORK #5 - MATH 4160

DUE: FRIDAY APRIL 4, 2003 AT 10:30AM

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups (maximum 3). You need only hand in one assignment per group, and write all names at the top.

Enumeration problems.

(1) Below are pictures of the 5 regular solids. Determine the orders of each of the symmetry groups by describing a procedure which picks an 'up face' and then one that picks the rotation of the solid that leaves the 'up face' fixed.



Note: the icosahedron has 20 faces, 30 edges and 12 vertices and the dodecahedron has 12 faces, 30 edges and 20 vertices.

(2) We say that a permutation π has a *descent* at postition k if $\pi(k) > \pi(k+1)$. How many permutations of n have only one descent and the descent is at position k where $1 \leq k \leq n$? How many permutations of n have at most one descent?

(3) (Bonus: grad students should do this) Let C_n represent the cyclic group of order n. For any number d, every abelian group of order d is isomorphic to $C_{r_1} \times C_{r_2} \times \cdots \times C_{r_k}$ for some sequence of r_i where $r_1 r_2 \cdots r_k = d$. By problem 1 in the next group of questions we know that $C_{r_1} \times C_{r_2} \cong C_{r_2} \times C_{r_1}$ so when considering an equivalence class of abelian groups, without loss of generality we can assume that $r_1 \geq r_2 \geq \cdots \geq r_k$. With this assumption one can show that if each r_i is a power of p for some prime p then $C_{r_1} \times C_{r_2} \times \cdots \times C_{r_k} \cong C_{s_1} \times C_{s_2} \times \cdots C_{s_\ell}$ if and only if $(r_1, r_2, \ldots, r_k) = (s_1, s_2, \ldots, s_\ell)$. How many equivalence classes of groups are there of order p^n where p is prime? (Suggestion: write out all possible abelian groups of order 3^2, 3^3, 3^4, 3^5 and try to make a connection with a combinatorial object we have seen before).

Groups and permuations:

(1) Prove that for any groups G and H that $G \times H$ is isomorphic to $H \times G$.

(2) Write down the multiplication table for the group A_4 of the 12 permutations of S_4 with even length (recall $length(\pi) = \#\{(i,j) : i < j \text{ and } \pi(i) > \pi(j)\}$). State how the multiplication table shows this is closed and every element has an inverse.
  (a) What is the maximum order of an element in the group?
  (b) How many elements are there of each order?
  (c) What is the largest proper subgroup of this group?
  (d) Show that the group is not abelian by producing two elements x and y such that $xy \neq yx$.
  (e) Is there a proper subgroup of this group which is not abelian?

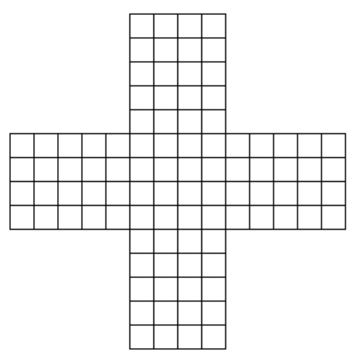
(3) (a) What is the size of the conjugacy class of elements with cycle type $(9,9,7,7,7,7,6,6,6,3)$ inside of S_{67}?
  (b) What is the order of each of the elements in this conjugacy class?
  (c) What is the largest conjugacy class in S_{47}?
  (d) What is the smallest conjugacy class in S_{47}?
  (e) What is the maximum order of an element in S_{47}?

Burnside's theorem and Pólya enumeration:

(1) How many different patterns can be formed by assembling 27 black and white cubes to form a $3 \times 3 \times 3$ cube? (consider only rotational symmetries of the cube).

(2) How many different patterns can be formed by assembling 27 different cubes such that 1 is black, 8 are red, 9 are white and 9 are blue into a $3 \times 3 \times 3$ cube.

(3) How many different patterns can be formed by coloring the following pattern with c different colors? Consider rotational and reflections of the figure.



(4) How many different ways are there of coloring the pattern above with 48 squares colored black and 48 of the squares colored white?