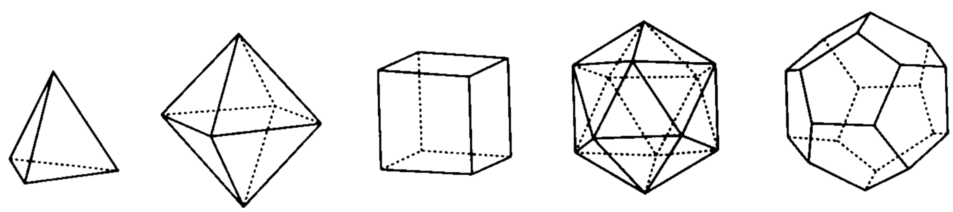
HOMEWORK #5 SOLUTIONS - MATH 4160

DUE: FRIDAY APRIL 4, 2003 AT 10:30AM

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups (maximum 3). You need only hand in one assignment per group, and write all names at the top.

Enumeration problems.

(1) Below are pictures of the 5 regular solids. Determine the orders of each of the symmetry groups by describing a procedure which picks an 'up face' and then one that picks the rotation of the solid that leaves the 'up face' fixed.



Note: the icosahedron has 20 faces, 30 edges and 12 vertices and the dodecahedron has 12 faces, 30 edges and 20 vertices.

Solution: The symmetry group of the solid above will be the set of rotations of the solid such that the orientation of the object looks the same before and after the rotation. That is, we look at each solid individually and start with one face up and one face towards us and then rotate the solid until we again have this configuration. How many different rotations are possible?

The tetrahedron has 4 faces. When we pick which one of the 4 is facing up, there are 3 choices for the face which is toward us. This means that there are 12 symmetries of the tetrahedron.

The cube has 6 faces. When we choose which one of the 6 is facing up, there are 4 possibilities for the one that is facing toward us. There are $24 = 6 \cdot 4$ symmetries of the cube.

The octahedron has 8 faces for the up face and 3 choices for the face which is toward us so again the symmetry group has $24 = 8 \cdot 3$ symmetries.

The icosahedron has 20 faces and each of these faces is adjacent to 3 others. Once we determine which of the 20 faces is our 'up' face, there are 3 choices for the face that is toward us and thus there are 60 symmetries of this object.

The dodecahedron has 12 faces and each of these faces is adjacent to 5 other faces. The symmetry group will have order $60 = 12 \cdot 5$ since once we choose the 'up' face, there are then 5 choices for the face which is toward us.

(2) We say that a permutation π has a *descent* at postition k if $\pi(k) > \pi(k+1)$. How many permutations of n have only one descent and the descent is at position k where $1 \leq k \leq n$? How many permutations of n have at most one descent?

Solution: A permutation with a descent at position k means that the permutation $\pi = \pi_1\pi_2\cdots\pi_n$ has the property that $\pi_1 < \pi_2 < \cdots < \pi_k$ and $\pi_k > \pi_{k+1}$ and $\pi_{k+1} < \pi_{k+2} < \cdots < \pi_n$. I can choose a subset of k numbers from a set of size n and let them be the numbers π_1 through π_k in increasing order and the remaining $n-k$ integers will be π_{k+1} through π_n in increasing order. There will be a descent at position k if and only if the subset of size k is not $\{1, 2, \ldots, k\}$ because in that case $\pi_k = k$ and $\pi_{k+1} = k+1$. Therefore there are $\binom{n}{k} - 1$ permutations with exactly one descent at position k.

The number of permutations with at most one descent will be the permutations with no descents (i.e. only the permutation $123\cdots n$) plus the number of permutations with a descent at position 1, or at position 2, or at position 3 etc. Therefore there are

$$1 + \binom{n}{1} - 1 + \binom{n}{2} - 1 + \cdots + \binom{n}{n-1} - 1 = \sum_{k=0}^{n} \binom{n}{k} - n = 2^n - n$$

The permutations with at most one descent are known as the Grassmannian permutations.

(3) (Bonus: grad students should do this) Let C_n represent the cyclic group of order n. For any number d, every abelian group of order d is isomorphic to $C_{r_1} \times C_{r_2} \times \cdots \times C_{r_k}$ for some sequence of r_i where $r_1 r_2 \cdots r_k = d$. By problem 1 in the next group of questions we know that $C_{r_1} \times C_{r_2} \cong C_{r_2} \times C_{r_1}$ so when considering an equivalence class of abelian groups, without loss of generality we can assume that $r_1 \geq r_2 \geq \cdots \geq r_k$. With this assumption one can show that if each r_i is a power of p for some prime p then $C_{r_1} \times C_{r_2} \times \cdots \times C_{r_k} \cong C_{s_1} \times C_{s_2} \times \cdots C_{s_\ell}$ if and only if $(r_1, r_2, \ldots, r_k) = (s_1, s_2, \ldots, s_\ell)$. How many equivalence classes of groups are there of order p^n where p is prime? (Suggestion: write out all possible abelian groups of order 3^2, 3^3, 3^4, 3^5 and try to make a connection with a combinatorial object we have seen before).

Solution: I suggested that you write out the possible groups for $p = 3$ and $n = 2, 3, 4, 5$. The answer is the same no matter what value of p you choose.

$$C_p \times C_p \qquad C_{p^2}$$

$$C_p \times C_p \times C_p \qquad C_{p^2} \times C_p \qquad C_{p^3}$$

$$C_p \times C_p \times C_p \times C_p \qquad C_{p^2} \times C_p \times C_p \qquad C_{p^2} \times C_{p^2} \qquad C_{p^3} \times C_p \qquad C_{p^4}$$

$$C_p \times C_p \times C_p \times C_p \times C_p \qquad C_{p^2} \times C_p \times C_p \times C_p \qquad C_{p^2} \times C_{p^2} \times C_p$$

$$C_{p^3} \times C_p \times C_p \qquad C_{p^3} \times C_{p^2} \qquad C_{p^4} \times C_p \qquad C_{p^5}$$

Notice that the number of classes of abelian groups of order p^n is equal to the number of partitions of n. That is, for every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ of n there is a group of order p^n of the form $C_{p^{\lambda_1}} \times C_{p^{\lambda_2}} \times \cdots \times C_{p^{\lambda_{\ell(\lambda)}}}$ and every abelian group of order p^n is isomorphic to a group of this form.

Groups and permuations:

(1) Prove that for any groups G and H that $G \times H$ is isomorphic to $H \times G$.

Solution: I didn't want you to bother with this question since we didn't end up discussing the definition of 'isomorphic' in class nor the product of groups. I assume that the grad students who were responsible for problem number (3) above are able to prove this if necessary.

(2) Write down the multiplication table for the group A_4 of the 12 permutations of S_4 with even length (recall $length(\pi) = \#\{(i,j) : i < j \text{ and } \pi(i) > \pi(j)\}$). State how the multiplication table shows this is closed and every element has an inverse.
   (a) What is the maximum order of an element in the group?
   (b) How many elements are there of each order?
   (c) What is the largest proper subgroup of this group?
   (d) Show that the group is not abelian by producing two elements x and y such that $xy \neq yx$.
   (e) Is there a proper subgroup of this group which is not abelian?

Solution: See the group table on the last page to answer these questions. Every row of this table consists only of elements whose cycle type is $(1,1,1,1), (3,1)$ or $(2,2)$.
   (a) Every element in this group is order either 1, 2 or 3. Therefore the maximum order of an element is 3.
   (b) There are 3 elements of order 2, 8 elements of order 3 and 1 element of order 1.
   (c) The largest proper subgroup of this group is $\{(1), (12)(34), (13)(24), (14)(23)\}$. This can be determined from looking at the table. Since the order of a subgroup must divide the order of the group the order of a proper subgroup of A_4 must be $1, 2, 3, 4$ or 6. If our subgroup was of order 6 then it must include the identity and at least two elements of order 3, say (abc) and (acb). We note that if it includes any of the elements of order two then by closure it will also include two more elements of order 3 since the product of an element of order 2 and one of order 3 is an element of order 3 and the inverse. If it includes four elements of order 3 then it includes at least 2 elements of order 2. Now this proves that the subgroup contains more than 6 elements so it must be the entire group (and hence is not 'proper').
   (d) The group is not abelian $(123) \circ (124) = (13)(24)$ and $(124) \circ (123) = (14)(23)$.
   (e) No. All proper subgroups are abelian. This is easy to check on the multiplication table since all the proper subgroups are of order $2, 3$ or 4.

(3) (a) What is the size of the conjugacy class of elements with cycle type $(9, 9, 7, 7, 7, 7, 6, 6, 6, 3)$ inside of S_{67}?
   (b) What is the order of each of the elements in this conjugacy class?
   (c) What is the largest conjugacy class in S_{47}?
   (d) What is the smallest conjugacy class in S_{47}?
   (e) What is the maximum order of an element in S_{47}?

Solution:

(a) The number of elements of cycle type $(9, 9, 7, 7, 7, 7, 6, 6, 6, 3)$ is given by the formula

$$\frac{67!}{9^2 \cdot 2! \cdot 7^4 \cdot 4! \cdot 6^3 \cdot 3! \cdot 3} \approx 1.004857133 \times 10^{84}$$

(b) The order of an element with this cycle type is $lcm(9, 7, 6, 3) = 126$.

(c) The largest conjugacy class will be one that minimizes the denominator of the formula for the size of the conjugacy class. That is we want $\prod_{k>0} k^{m_k(\lambda)} m_k(\lambda)!$ as small as possible for all partitions λ of size 47. This happens for $\lambda = (46, 1)$ and the conjugacy class will have $47 \cdot 45!$ elements in it.

(d) The smallest conjugacy class of S_{47} consists of just the identity.

(e) The order of an element with cycle type $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is $lcm(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$. To maximize this we should choose each of the λ_i to be relatively prime to each other. A starting point for a search of a partition which maximizes this quantity is to consider $47 = 2 + 3 + 5 + 7 + 11 + 13 + 6$ (that is the sum of the first 6 prime numbers and we are left with 6 left over). The elements of cycle type $(13, 11, 8, 7, 5, 3)$ have order 120120 and this is the maximum possible order of an element of S_{47}.

Burnside's theorem and Pólya enumeration:

(1) How many different patterns can be formed by assembling 27 black and white cubes to form a $3 \times 3 \times 3$ cube? (consider only rotational symmetries of the cube).

Solution: I have included on the web page a few paragraphs on how to group together all of the rotations of the cube. See the note at `http://garsia.math.yorku.ca/~zabrocki/math4160w03/cubesyms/`.

Every coloring of the cubes is fixed by the identity element and there are 2^{26} such colorings (note: we need not color the center cube since it is hidden by the others).

There are 6 rotations of $\pm 90°$ leaving two faces in place. The number of colorings of the cube which are fixed by these permutations is 2^8.

There are 3 rotations of $180°$ leaving two faces in place. The number of colorings of the 26 cubes which are fixed by these permutations is 2^{14}.

There are 6 rotations by $180°$ of cube leaving 2 edges in the same place. There are again 2^{14} colorings of the cubes which are fixed by these permutations.

There remain 8 rotations of the cube leaving two corners fixed and 2^{10} ways of coloring the cubes such that they are fixed by these permutations.

This means that the total number of patterns (using Burnside's theorem) is

$$\frac{1}{24}\left(2^{26} + 6 \cdot 2^8 + 9 \cdot 2^{14} + 8 \cdot 2^{10}\right) = 2802752$$

(2) How many different patterns can be formed by assembling 27 different cubes such that 1 is black, 8 are red, 9 are white and 9 are blue into a $3 \times 3 \times 3$ cube.

Solution: The cycle index polynomial for this group action is given by

$$\frac{1}{24}\left(x_1^{27} + 9x_1^3 x_2^{12} + 8x_1^3 x_3^8 + 6x_1^3 x_4^6\right).$$

This is found by looking at the $3\times3\times3$ cube and observing the orbits of each of the blocks. For any of the permutations except the identity there are exactly 3 fixed points and the other 24 cubes are permuted amongst themselves in orbits of size 2, 3 or 4.

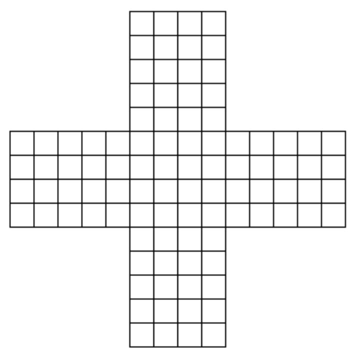
Now, to compute the coefficient of $B^1r^8w^9b^9$ in the polynomial above with x_k replaced by $B^k + r^k + w^k + b^k$. This means we want the coefficient of $B^1r^8w^9b^9$ in

$$\frac{1}{24}((B{+}r + w + b)^{27} + 9(B + r + w + b)^3(B^2 + r^2 + w^2 + b^2)^{12}+$$
$$8(B + r + w + b)^3(B^3 + r^3 + w^3 + b^3)^8 + 6(B + r + w + b)^3(B^4 + r^4 + w^4 + b^4)^6).$$

This is equal to

$$\frac{1}{24}\left(\frac{27!}{8!\ 9!\ 9!} + 9\cdot 6\cdot \frac{12!}{4!\ 4!\ 4!} + 8\cdot 3\cdot \frac{8!}{2!\ 3!\ 3!} + 6\cdot 6\cdot \frac{6!}{2!\ 2!\ 2!}\right) = 85452615470.$$

(3) How many different patterns can be formed by coloring the following pattern with c different colors? Consider rotational and reflections of the figure.



Solution: There are 8 elements in the symmetry group for this pattern (just as for the square), 4 flips and 3 rotations and the identity. Every square is in an orbit of 4 under rotations by $\pm 90°$ and each of those $96/4 = 24$ orbits may be colored independently. A rotation by $180°$, or a flip about the horizontal or vertical of the diagram groups the squares into pairs so that there are $96/2 = 48$ ways of coloring the diagram so that it is fixed by these elements. When the diagram is flipped about either of the diagaonals, 4 squares stay fixed and the other 92 squares are paired up so that there are $4+92/2 = 50$ ways of coloring the diagram. This means that there are

$$\frac{1}{8}\left(c^{96} + 2c^{24} + 3c^{48} + 2c^{50}\right)$$

ways of coloring this diagram with c colors.

(4) How many different ways are there of coloring the pattern above with 48 squares colored black and 48 of the squares colored white?

Solution: The cycle index polynomial of this group action is

$$\frac{1}{8}\left(x_1^{96} + 2x_4^{24} + 3x_2^{48} + 2x_1^4x_2^{46}\right).$$

Look at the figure and each of the group elements...if you don't know how to do this part, come talk to me.

Now, replace x_k with $b^k + w^k$ and take the coefficient of $b^{48}w^{48}$ in this polynomial. The polynomial is

$$\frac{1}{8}\left((b+w)^{96} + 2(b^4+w^4)^{24} + 3(b^2+w^2)^{48} + 2(b+w)^4(b^2+w^2)^{46}\right)$$

The coefficient of $b^{48}w^{48}$ is

$$\begin{aligned}\frac{1}{8}&\left(\frac{96!}{48!\,48!} + 2\frac{24!}{12!\,12!} + 3\frac{48!}{24!\,24!} + 2\left(\frac{46!}{22!\,24!} + 6\frac{46!}{23!\,23!} + \frac{46!}{22!\,24!}\right)\right)\\ &= 8043833767333157517364059 64\end{aligned}$$

GROUP TABLE FOR A_4

GROUPS AND PERMUTATIONS PROBLEM (2)

	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(132)	(124)	(142)	(134)	(143)	(234)	(243)
(1)	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(132)	(124)	(142)	(134)	(143)	(234)	(243)
(12)(34)	(12)(34)	(1)	(14)(23)	(13)(24)	(243)	(143)	(234)	(134)	(142)	(132)	(124)	(123)
(13)(24)	(13)(24)	(14)(23)	(1)	(12)(34)	(142)	(234)	(143)	(123)	(243)	(124)	(132)	(134)
(14)(23)	(14)(23)	(13)(24)	(12)(34)	(1)	(134)	(124)	(132)	(243)	(123)	(234)	(143)	(142)
(123)	(123)	(134)	(243)	(142)	(132)	(1)	(13)(24)	(143)	(234)	(14)(23)	(12)(34)	(124)
(132)	(132)	(234)	(124)	(143)	(1)	(123)	(243)	(14)(23)	(12)(34)	(142)	(134)	(13)(24)
(124)	(124)	(143)	(132)	(234)	(14)(23)	(134)	(142)	(1)	(13)(24)	(243)	(123)	(12)(34)
(142)	(142)	(243)	(134)	(123)	(234)	(13)(24)	(1)	(124)	(132)	(12)(34)	(14)(23)	(143)
(134)	(134)	(123)	(142)	(243)	(124)	(14)(23)	(12)(34)	(234)	(143)	(1)	(13)(24)	(132)
(143)	(143)	(124)	(234)	(132)	(12)(34)	(243)	(123)	(13)(24)	(1)	(134)	(142)	(14)(23)
(234)	(234)	(132)	(143)	(124)	(13)(24)	(142)	(134)	(12)(34)	(14)(23)	(123)	(243)	(1)
(243)	(243)	(142)	(123)	(134)	(143)	(12)(34)	(14)(23)	(132)	(124)	(13)(24)	(1)	(234)