GENERATING FUNCTIONS PRACTICE - MATH 4160

Find the generating function for the each of the following sets of partitions described below.

(1) Partitions with only even parts and the length is ≤ 6.
(2) Partitions with only even parts and the width is ≤ 6.
(3) Partitions with distinct even parts of width ≤ 6.
(4) Partitions with the number of parts of size k is always a multiple of k for all $k \geq 1$.
(5) Partitions which are not of length exactly equal to 2.
(6) Partitions with the number of parts of size k is always $\leq k$ for all $k \geq 1$.
(7) Partitions with width less than or equal to n and odd parts are repeated at most 4 times and even parts are repeated at most twice.
(8) Partitions of height less than or equal to n where every value of k for $1 \leq k \leq n$ appears as a part exactly once.
(9) Partitions with distinct odd parts
(10) Self conjugate partitions (partitions with the property that if the diagram is flipped about the $x = y$ line then the picture stays the same).
(11) Partitions of the form (a, b) where $a > b$.
(12) Odd partitions which are not self-conjugate.
(13) Partitions of width less than or equal to 3 or of length less than or equal to 2 (hint: Inclusion-Exclusion might work on this)
(14) Partitions with length $\leq k$ and either there is one even part OR there are two even parts and at least one odd part.
(15) (**) Partitions which fit inside of an $n \times k$ rectangle.

(**) possibly difficult but it has a simple answer. Hint: Note the generating function is finite. Recall that the number of partitions which fit inside of an $n \times k$ rectangle is $\binom{n+k}{k}$ and so if we evaluate the generating function at $x = 1$ then the answer must be this number.

Give a description of a set of combinatorial objects that is counted by the following formulas. The formulas include the variables a_n and b_n where n is an integer greater than or equal to 0. We will take a_n to represent the number of *widgets* of size n and b_n to represent the number of *doodles* of size n. There are many answers to these problems but someone should be able to read your description and recover the formula.

(1) n^2

(2) $n(n-1)/2$

(3) 2^n

(4) $n!$

(5) na_k

(6) $a_0 a_1 a_2 \cdots a_k$

(7) a_k^2

(8) $\sum_{k=0}^{n} a_k^2$

(9) a_k^n

(10) $\sum_{k=0}^{r} a_k$

(11) $\sum_{k=0}^{r} a_k^n$

(12) $\left(\sum_{k=0}^{r} a_k\right)^n$

(13) $\binom{n}{k} a_k b_{n-k}$

(14) $a_k b_{n-k}$

(15) $\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$

(16) $\sum_{k=0}^{n} a_k b_{n-k}$