PRACTICE FOR MIDTERM - MATH 4160

THIS IS NOT DUE, BUT YOU HAVE A SECOND MIDTERM ON FRIDAY, MARCH 7 AND SO IT IS A GOOD IDEA TO TRY THESE PROBLEMS

(1) Give a generating function for partitions which satisfy the following properties:
  (a) have only even parts
  (b) have even parts where all parts are distinct
  (c) have even or odd parts but the odd parts may only occur once
  (d) have exactly 4 parts and all of them are odd
  (e) only odd parts and each part is less than or equal to $2k+1$
  (f) all parts are less than or equal to k and no part is repeated more than 2 times
  (g) even parts and the length of the partition is less than or equal to k

(2) Assume that $A(x) = \sum_{n\geq 0} a_n x^n$ is the generating function for the sequence $(a_0, a_1, a_2, \ldots)$ where a_i is the number of *widgets* of size i. Explain in words what the coefficient of x^n in the generating function $A(x)\frac{1}{1-x}$ represents.

(3) Fix $n > 0$ and find a simple expression for the generating function for the numbers $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ (recall that $\binom{n}{k} = 0$ if $k > n$).

(4) Fix a number k and find a simple expression for the generating function for the numbers $\binom{0}{k}, \binom{1}{k}, \binom{2}{k}, \binom{3}{k}, \ldots$.

(5) Find the generating function for the sequences that satisfy the following recurrences:
  (a) $a_0 = 0$, $a_1 = 1$, $a_n = 3a_{n-1} - a_{n-2}$
  (b) $r_0 = 1$, $r_1 = 1$, $r_n = r_{n-1} - r_{n-2} + (n^2 - n)/2$
  (c) $M_0 = 0$, $M_1 = 1$, $M_n = 3M_{n-1} - 2M_{n-2}$ (Mersenne numbers)
  (d) $m_0 = 1$, $m_1 = 1$, $m_n = m_{n-1} + \sum_{i=0}^{n-2} m_i m_{n-i}$ (Motzkin numbers)