

X_1 = heads or tails penny

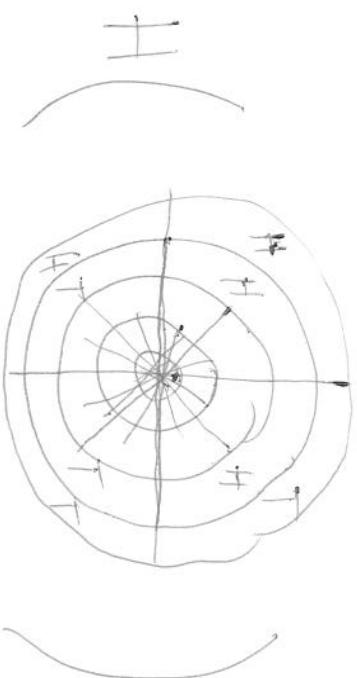
X_2 = heads or tails nickel

X_{25} = heads or tails quarter

$X_{100} = \text{H/t loonie}$

$X_{200} = \text{H/t toonie}$

$$H(X_1, X_5, X_{25}, X_{100}, X_{200})$$

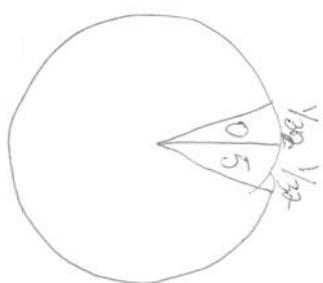


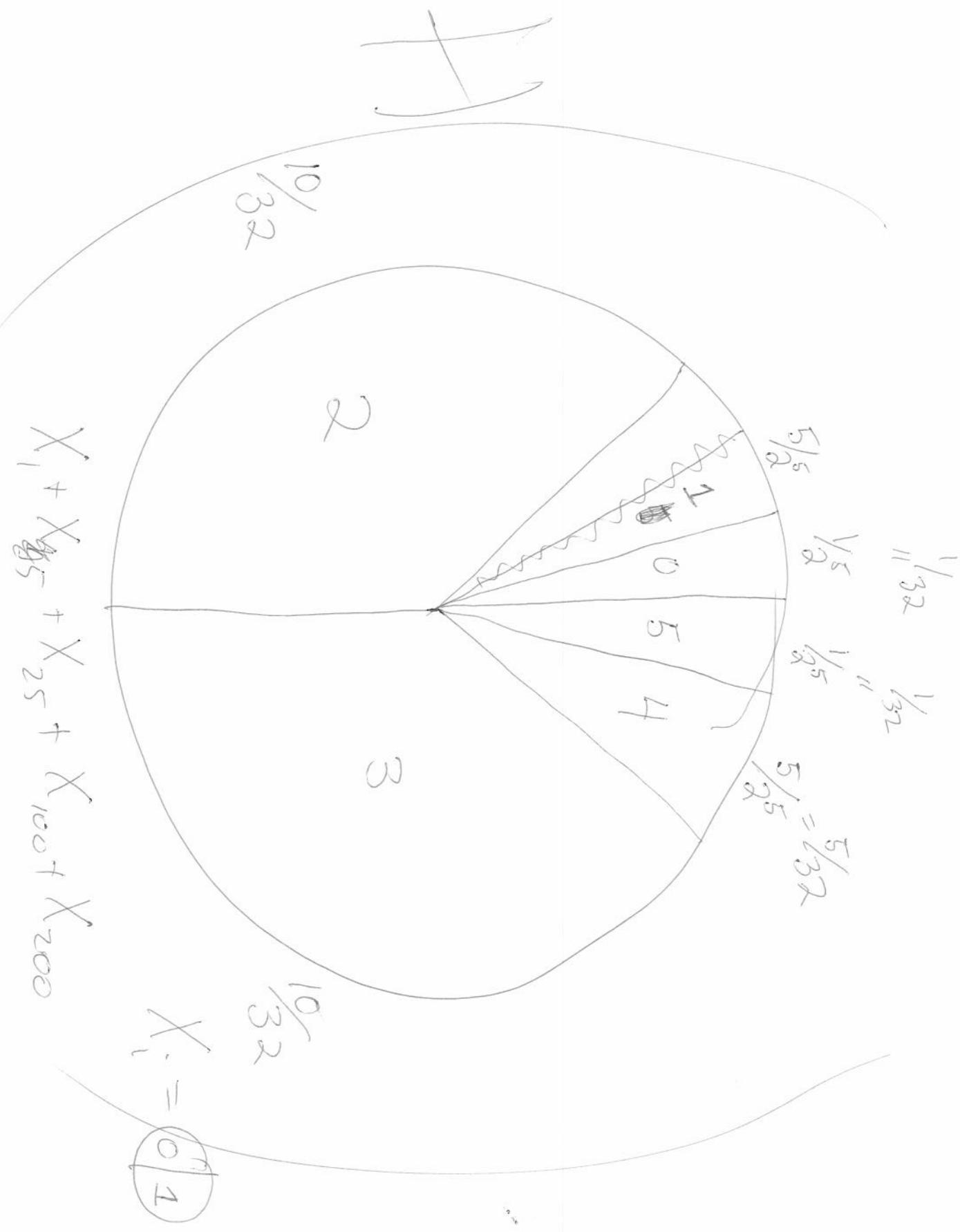
$$H\left(\left(\frac{H}{T}\right) + H\left(\frac{H}{T}\right) + H\left(\frac{H}{T}\right) + H\left(\frac{H}{T}\right)\right)$$

$$\equiv 5$$

$$H(X_1 + X_5 + X_{25} + X_{100} + X_{200})$$

$H = 1 \quad T = 0$





$$H\left(X_1 + X_5 + X_{25} + X_{100} + X_{200}\right) = \frac{1}{32} \log_2\left(\frac{32}{1}\right) + \frac{5}{32} \log_2\left(\frac{32}{5}\right) + \frac{10}{32} \log_2\left(\frac{32}{10}\right)$$

$$+ \frac{10}{32} \log_2\left(\frac{32}{10}\right) + \frac{5}{32} \log_2\left(\frac{32}{5}\right) + \frac{1}{32} \log_2\left(\frac{32}{1}\right)$$

estimate

$$\log_2 32 = 5$$

$$\log_2 5 \approx 2.3$$

$$\log_2 10 = \log_2 2 + \log_2 5 \\ \approx 1 + 2.3 = 3.3$$

$$\approx \left(\frac{1}{32} \cdot 5 + \frac{5}{32} (5 - 2.3) + \frac{10}{32} \log_2 (5 - 3.3) \right) \cdot 2$$

$$= 2 \cdot \left(\frac{5 + 2.5 + 50 - 10.5 - 3.3}{32} \right) = 2 \cdot \left(\frac{80 - 41.5}{32} \right) = 2 \cdot \frac{38.5}{32}$$

$$\leq 2.5$$

$$H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \log_2 3$$

$$= \frac{2}{8} - \frac{3}{8} \log_2 3$$

$$= \frac{3}{8}(3 - \log_2 3) + \frac{1}{8} \cdot 3 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2$$

(a) Calculate $H[X]$. ~~$= \frac{3}{8} \log_2 \left(\frac{8}{3}\right) + \frac{1}{8} \log_2 8 + \frac{1}{4} \log_2 \frac{4}{3}$~~

registers needed to store Z .

$$= H(Z)$$

(c) Calculate the uncertainty of Z given that

$$X = 0.$$

$$H(Z|X=0) = \frac{2}{3} \log_2 \left(\frac{3}{2}\right) + \frac{1}{3} \log_2 (3)$$

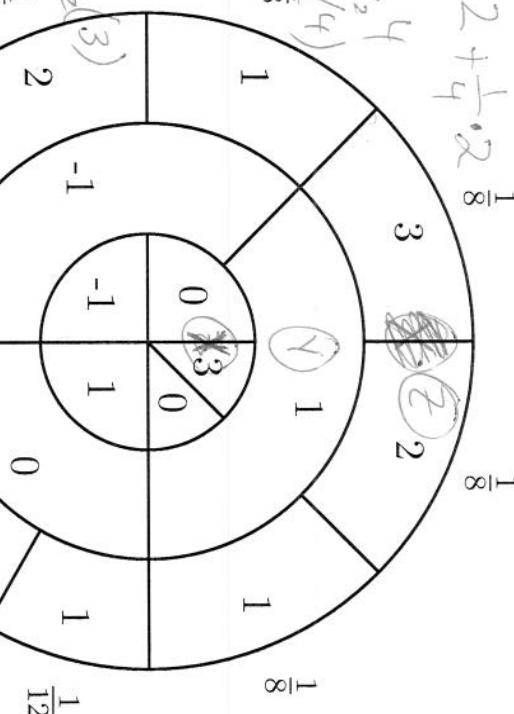
(d) Calculate $H[X|Y, Z]$.

~~X is independent on $Y \neq Z$~~

(e) Calculate $H[Z|Y]$.

$$= H(Z|Y=1) \cdot P(Y=1) + H(Z|Y=0) \cdot P(Y=0) + H(Z|Y=-1) \cdot P(Y=-1)$$

$$= \log_2 (3) \cdot \frac{3}{8} + \frac{1}{4} \log_2 (3) + \frac{3}{8} \log_2 (3) = \log_2 (3)$$



$Y \neq Z$ are independent

$$P(Z=1 | X=0) = \frac{P(Z=1 \wedge X=0)}{P(X=0)} = \frac{\frac{2}{8}}{\frac{3}{8}} = \frac{2}{3}$$

$$P(Z=3 | X=0) = \frac{1}{3}$$

$H(Z) = \text{amount of information learned when told } Z$
 $H(Z|Y) = \text{amount of info learned when told } Z$ given that I know Y

$$\begin{aligned}
 H(Z|X) &= H(Z|X=0) \cdot P(X=0) + H(Z|X=3) \cdot P(X=3) \\
 &\quad + H(Z|X=1) \cdot P(X=1) + H(Z|X=-1) \cdot P(X=-1) \\
 &= \left(\log_2 3 - \frac{2}{3}\right) \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} \\
 &\quad + \frac{1}{4} \log_2 3 + \frac{1}{4} \cdot 1
 \end{aligned}$$

$$= \frac{3}{8} \log_2 3 = \cancel{\frac{1}{4}} + \frac{1}{4} \log_2 3 + \cancel{\frac{1}{4}}$$

$$= \frac{5}{8} \log_2 (3)$$

$$H(Z)$$

$= \log_2 (3)$

when
Z and X
are independent

$=$ when
Z depends
on X

$$H(Z) \geq H(Z|X) \geq 0$$

for any random variable X

Theorem 2 For any two random variables X and Y we always have

$$H(X|Y) \leq H(X) \quad (1)$$

and equality holds if and only if X and Y are independent.

Proof. From our definitions we get

$$\begin{aligned}
 H(X|Y) &= \sum_b P[Y = b] H(X|Y = b) \\
 &= \sum_b P[Y = b] \sum_a P[X = a | Y = b] \log_2 \frac{1}{P[X = a | Y = b]} \\
 &= \sum_b P[Y = b] \sum_a \frac{P[X = a, Y = b]}{P[Y = b]} \log_2 \frac{1}{P[X = a | Y = b]} \\
 &= \sum_b \sum_a \underbrace{P[X = a, Y = b]}_{P[X = a | Y = b]} \log_2 \frac{1}{P[X = a | Y = b]} \\
 &= \sum_a P[X = a] \sum_b P[Y = b | X = a] \log_2 \frac{1}{P[X = a | Y = b]} \\
 &\stackrel{(2)}{\leq} \sum_a P[X = a] \log_2 \frac{1}{P[X = a]} = H(X)
 \end{aligned}$$

$$\left(\sum_i m_i x_i, \log_2 \left(\sum_i m_i x_i \right) \right)$$

$$\left(\sum_i m_i x_i, \sum_i m_i \log_2(x_i) \right)$$

take any
 $\sum m_i = 1$
 $m_i \geq 0$

Conclude : $\sum_i m_i \log_2(x_i) \leq \log_2 \left(\sum_i m_i x_i \right)$

