EXPERIMENT, RANDOM VARIABLES: This refers to an activity, not necessarily scientific, which involves the production of data some of which are “random”. We denote an experiment by $\mathbb{E}$ and the data by X, Y , Z, ... The latter are usually referred to as the RANDOM VARIABLES associated with $\mathbb{E}$.

RANDOM, SAMPLE SPACE, PROBABILITIES: We use the word RANDOM whenever the data X, Y , Z, . . . we are studying are produced by such an intricate mechanism that all we know about them is

(1) The range of possible values that X, Y , Z, . . . may take. This range is usually referred to as the SAMPLE SPACE and denoted by the symbol Ω.

(2) Certain positive numbers called PROBABILITIES which numerically express our “confidence” that X, Y , Z, . . . fall in chosen subsets of the sample space Ω.

ELEMENTARY OUTCOME, SAMPLE POINT: An individual outcome of the experiment E is usually referred to as an ELEMENTARY OUTCOME or SAMPLE POINT. Mathematically this is just an element of the sample space Ω.

EVENT: Mathematically an EVENT is just a subset of Ω. We say that $\mathtt{E}$ "resulted in the event A" or that "A has occurred" if the outcome falls in the subset A.

FIELD OF EVENTS: The collection of events associated with our experiment $\mathtt{E}$ is usually denoted by $\mathtt{F}$. In other words, $\mathtt{F}$ denotes the collection of subsets of the sample space Ω that are of special interest in our study. For mathematical reasons $\mathtt{F}$ is assumed to be closed under the set operations of intersection, union and complementation. The two subsets {} and Ω are always included in $\mathtt{F}$.

PROBABILITY MEASURE: Our experiment $\mathbb{E}$ associates to each event A of F a number P[A] in the interval [0, 1] which is reflects our confidence that the outcome falls in A. We refer to P[A] as the “probability of A.” Note that we should have P[Ω] = 1 and that if A and B are mutually exclusive events then

$$P[A \cup B] = P[A] + P[B]$$

A set function with these properties is usually referred to as a PROBABILITY MEASURE.

EXPECTATION OF A RANDOM VARIABLE: Any function of the outcome of our experiment can be referred to as a RANDOM VARIABLE. Mathematically, a random variable is simply a function on the sample space. If the events $A_1, A_2, \ldots, A_k$ are mutually exclusive and decompose Ω , and the random variable X takes the value x_i when A_i occurs then the expression

$$E[X] = x_1 P[A_1] + x_2 P[A_2] + \cdots + x_k P[A_k]$$

is referred to as the EXPECTATION OF X . If we repeat E a very large number of times, and average out the successive values of X we get, then we should expect the resulting average to be close to E[X].

CONDITIONAL PROBABILITY: If A and B are events the ratio

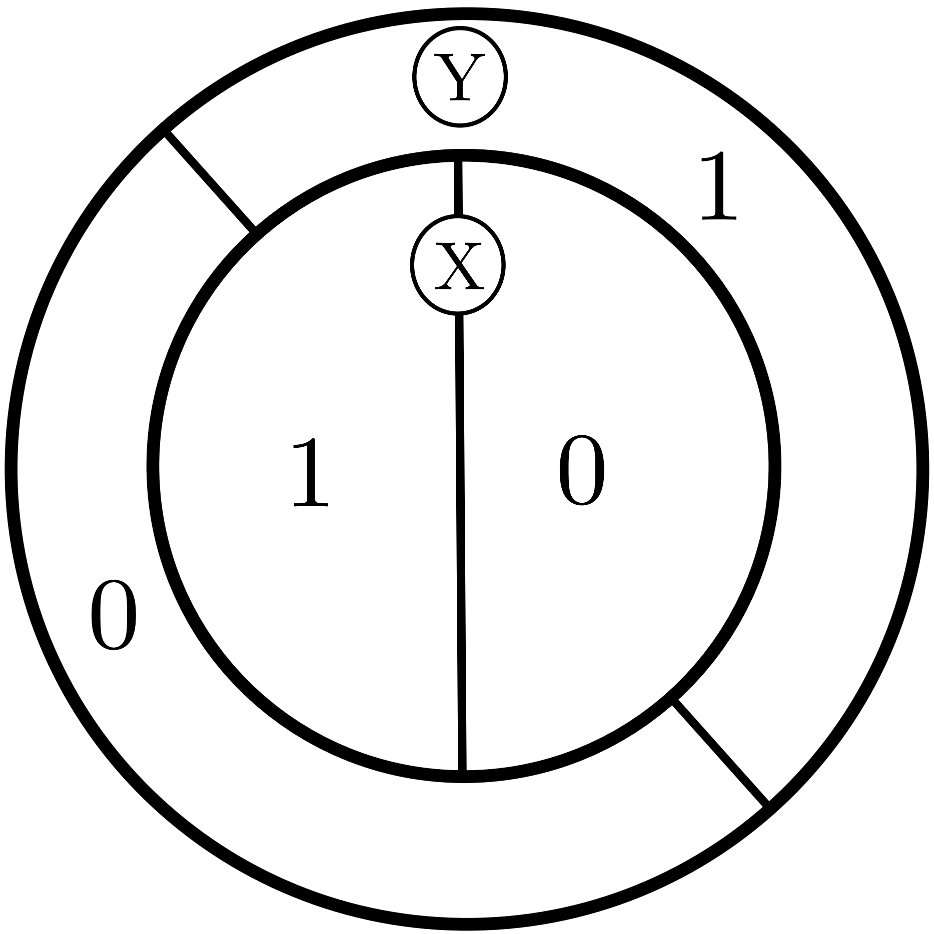
$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

is usually referred to as the CONDITIONAL PROBABILITY OF A GIVEN B. The concept arises as follows. Given the event B we can construct a new experiment E_B by carrying out E and recording its outcome only when it falls in B. We can argue that the probability of A under E_B will be the expression above where $P[A \cap B]$ and $P[B]$ are the probabilities of $A \cap B$ and B under E. We shall refer to E_B as E CRIPPLED by B.

CONDITIONAL EXPECTATION OF A RANDOM VARIABLE: Given an event B, if we carry out the crippled experiment E_B instead of E , then all the probabilities change and so do all expectations. If X is a random variable and the events $A_1, A_2, \ldots, A_k$ decompose Ω as before then expression

$$E[X|B] = x_1 P[A_1|B] + x_2 P[A_2|B] + \cdots + x_k P[A_k|B]$$

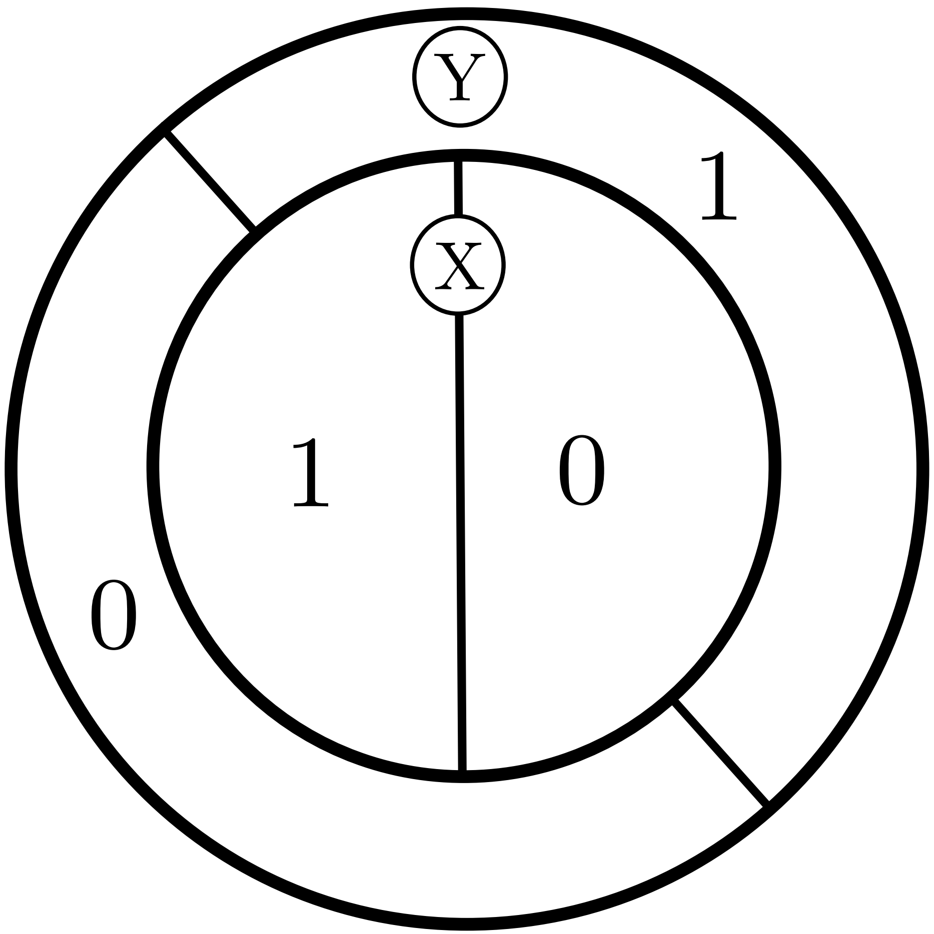
gives the expected value of X under E_B. We refer to it as the CONDITIONAL EXPECTATION OF X GIVEN B.



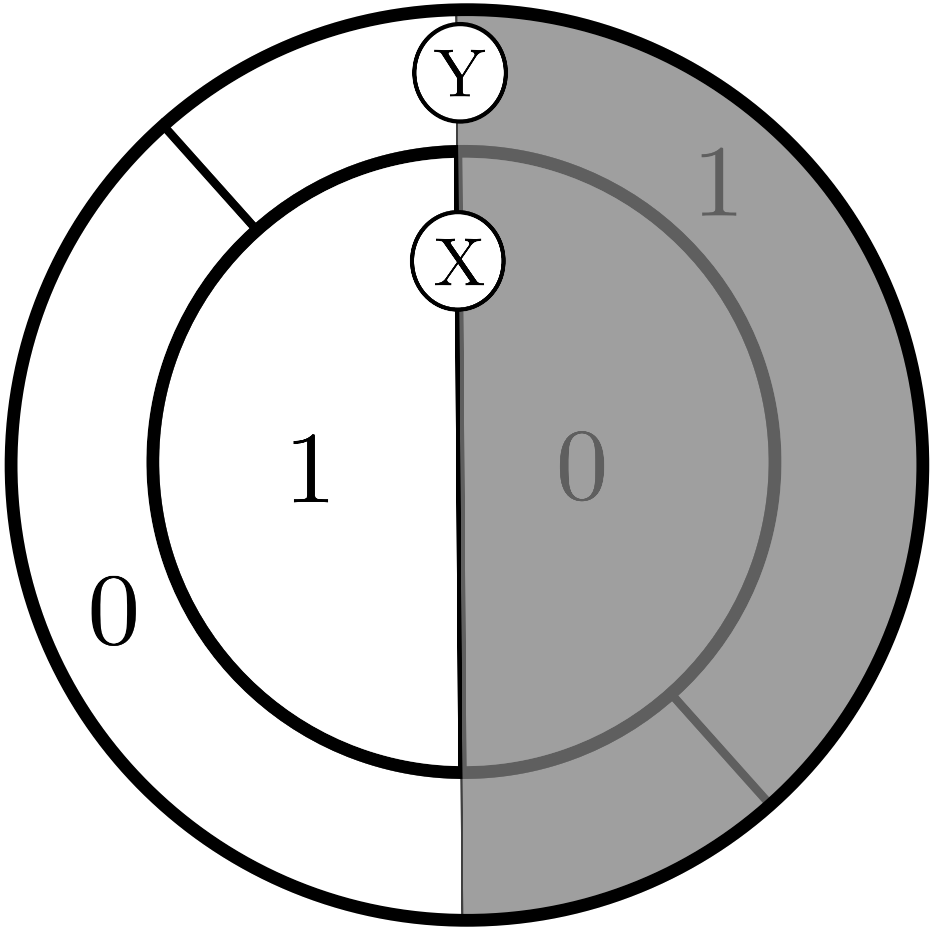
$$P(X{=}0) = P(X{=}1) = P(Y{=}0) = P(Y{=}1) = 1/2$$

$$P(X{=}1 \;\&\; Y{=}1) = P(X{=}0 \;\&\; Y{=}0) = 1/8$$

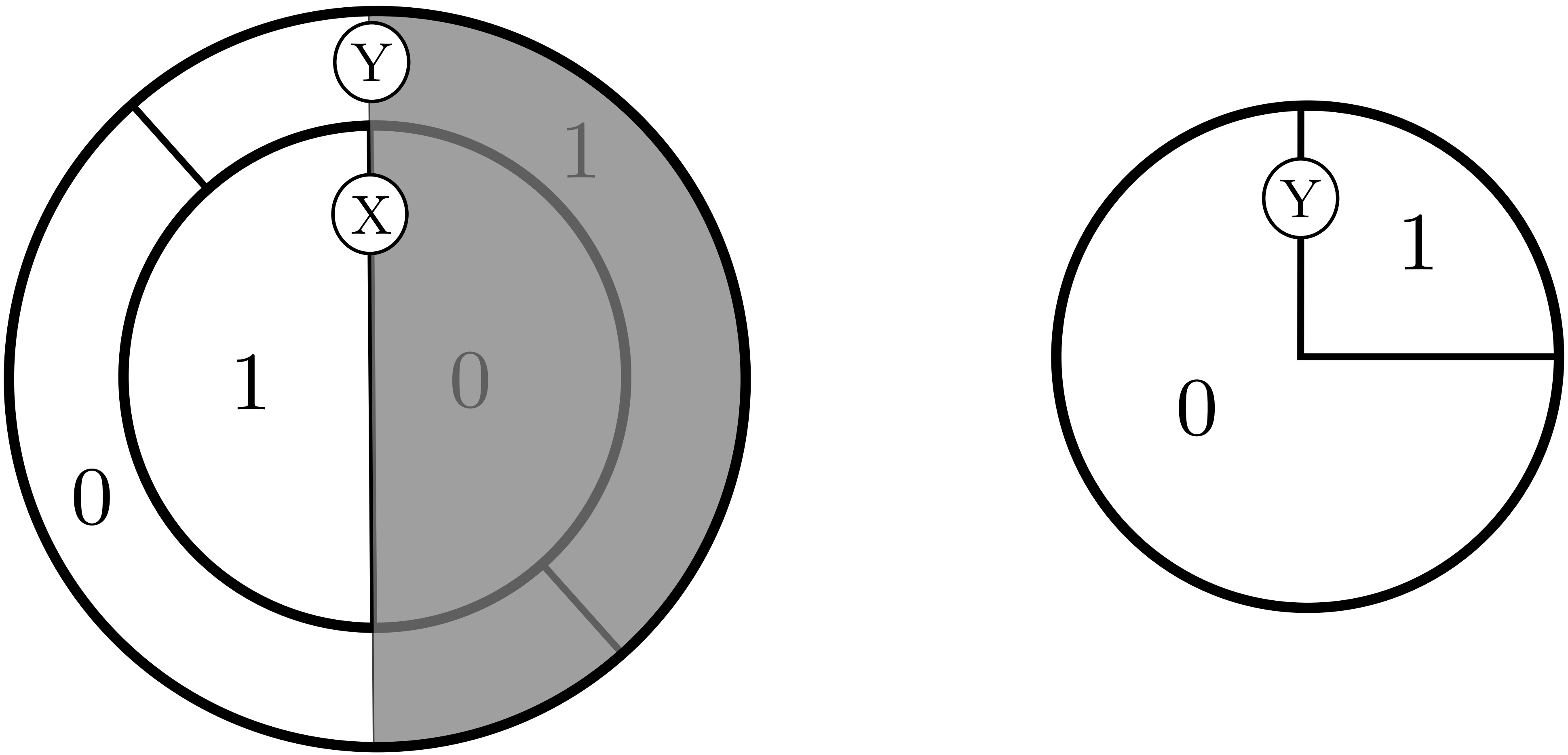
$$P(X{=}0 \;\&\; Y{=}1) = P(X{=}1 \;\&\; Y{=}0) = 3/8$$



P(Y=0 | X=1) =



$P(Y{=}0 \mid X{=}1\ ) =$



$$P(Y{=}0 \mid X{=}1\ ) = \frac{P(Y{=}0\ \&\ X{=}1)}{P(X{=}1)} = \frac{3/8}{1/2} = 3/4$$

DEPENDENCE: The random variable Y is said to be DEPENDENT upon the random variable X if and only if Y is a function of X . Similarly we say that Y is dependent upon X_1 , X_2 , . . . , X_n if for some function $f(x_1, x_2, \ldots, x_n)$ we have

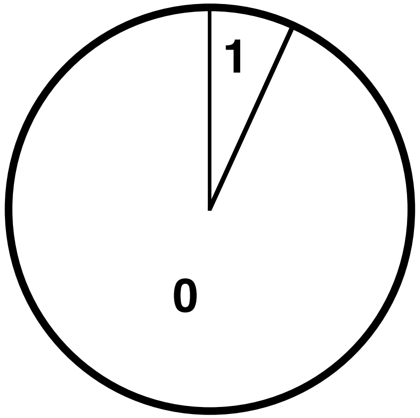
$$Y = f(X_1, X_2, \ldots, X_n)$$

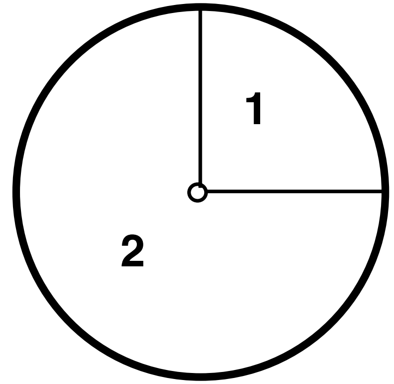
INDEPENDENCE: In probability theory, “independence” is not the negation of “dependence” We say that X is “independent” of Y only if knowing the value of Y “doesn’t change our uncertainty” about X. More precisely, if we cripple our experiment E by any of the events [Y = b] the probabilities of all the events [X = a] do not change. Mathematically this is translated in the conditions that for all choices of a and b

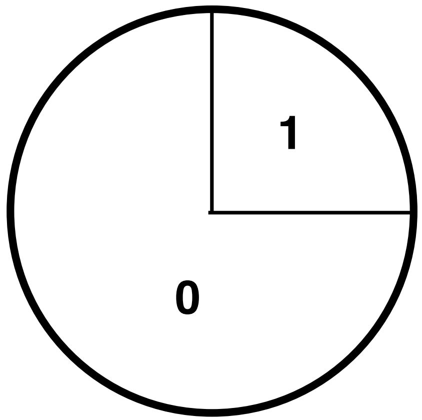
$$P(X = a | Y = b) = P(X = a)$$

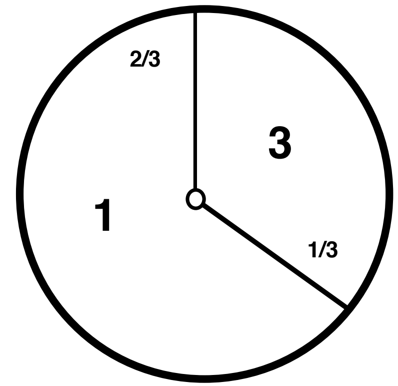
this simply means that

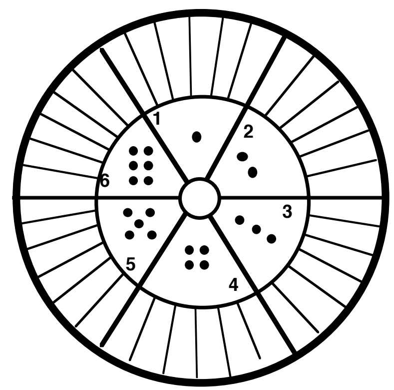
$$P(X = a \text{ and } Y = b) = P(X = a)P(Y = b)$$

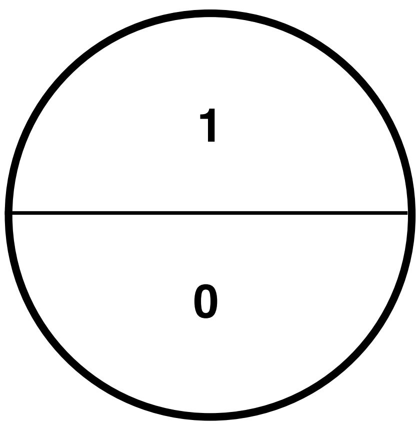

1
0

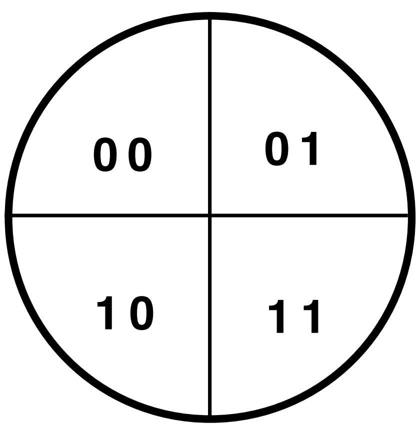

1
2

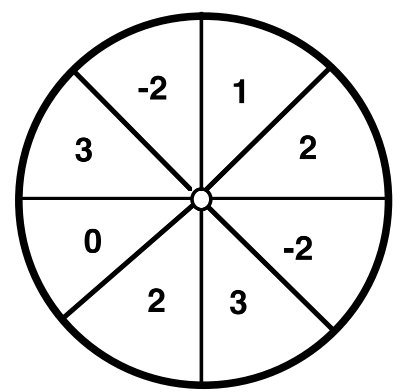

1
0


2/3
3
1
1/3


1
2
6
3
5
4


1
0


0 0
0 1
1 0
1 1

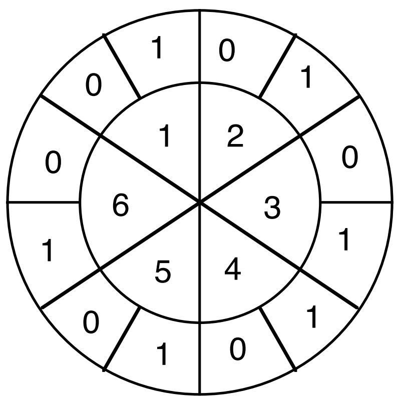

-2
1
3
2
0
-2
2
3

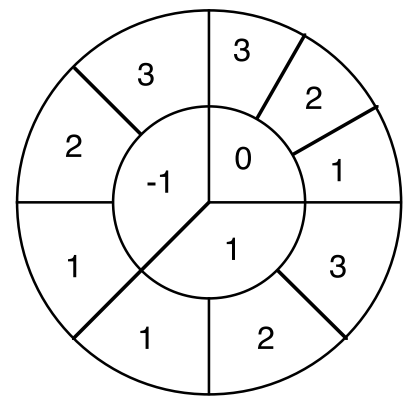
X is independent of Y if $P(X = a | Y = b) = P(X = a)$

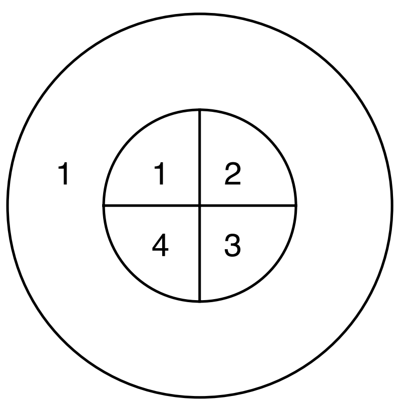
or $P(X = a \text{ and } Y = b) = P(X = a)P(Y = b)$

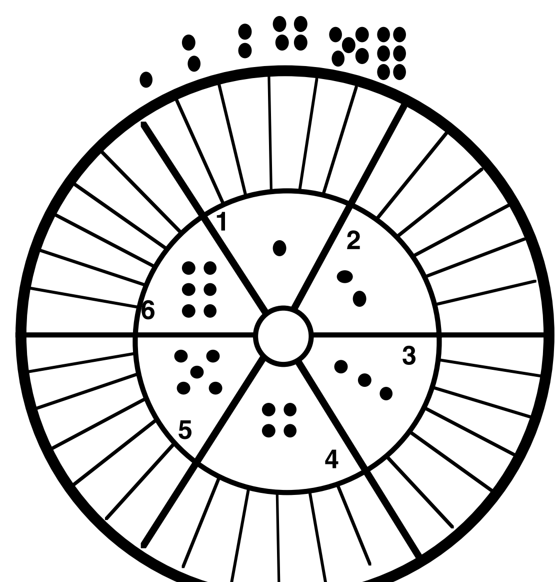
or knowing the value of Y does not change the probabilities of X

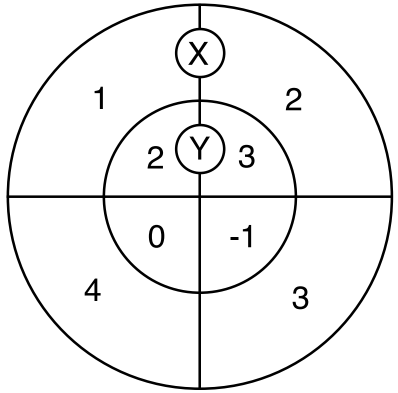
If X is independent of Y, then Y is independent of X.

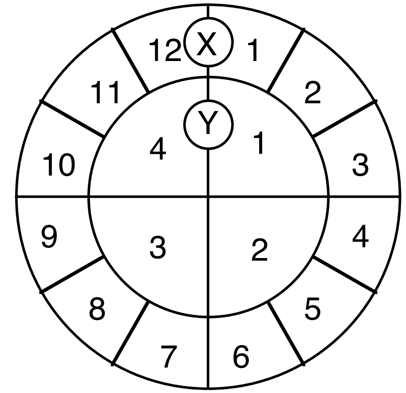


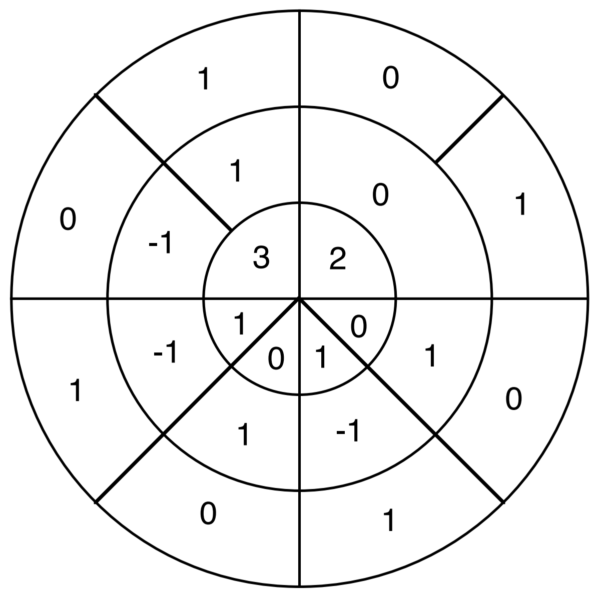












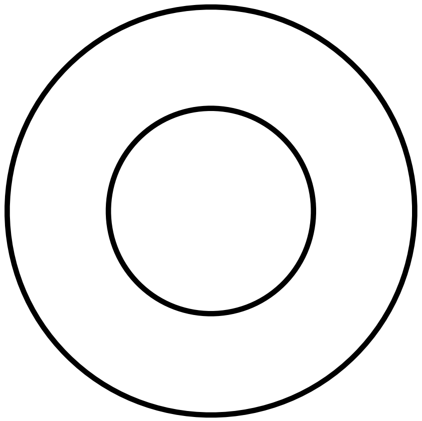
X is dependent on Y if X is a function of Y

that is, knowing the value of Y determines the value of X

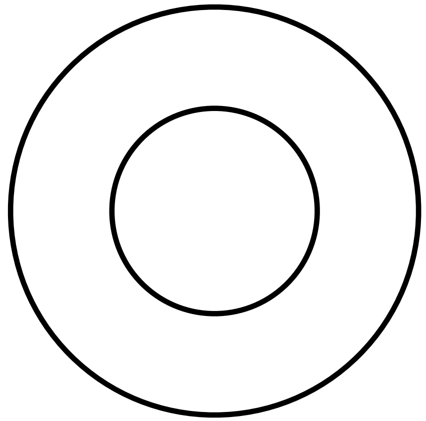
"X is dependent on Y" and "X is independent of Y" are not opposite statements of each other, rather they are on opposite sides of a spectrum of possibilities.

"X is not dependent on Y" does not mean "X is independent of Y"

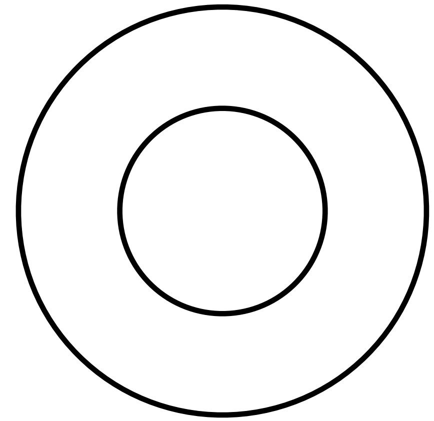
X is independent of Y
X is dependent on Y
Y is dependent on X



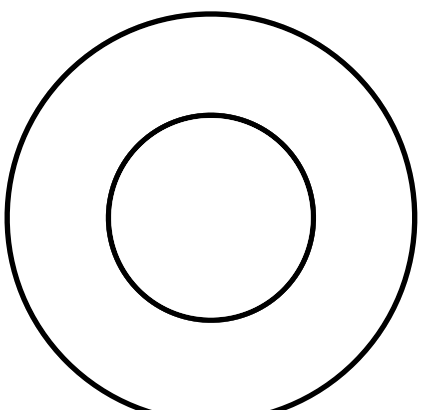
X is independent of Y
X is not dependent on Y
Y is dependent on X



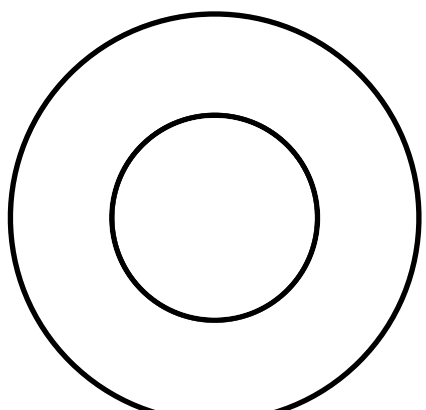
X is independent of Y
X is not dependent on Y
Y is not dependent on X



X is not independent of Y
X is dependent on Y
Y is dependent on X



X is not independent of Y
X is not dependent on Y
Y is dependent on X



X is not independent of Y
X is not dependent on Y
Y is not dependent on X