The Canonical Roulette and the game of Craps

1 The Canonical Roulette

To have a random number generator we shall imagine an ideal device that will be responsible for generating random numbers.

Imagine that it consists of a solid wheel whose outer circumference is of unit length (say one meter). This wheel has a ball bearing joint at its center, with an axle fastened to it which permits the wheel to rotate freely around its axis of revolution.

We shall also suppose that the axle is fastened perpendicularly upon a perfectly horizontal table. Furthermore, the friction on the bearing is to be so minimal that even the slightest initial spin causes the wheel to undergo several revolutions before coming to a stop.

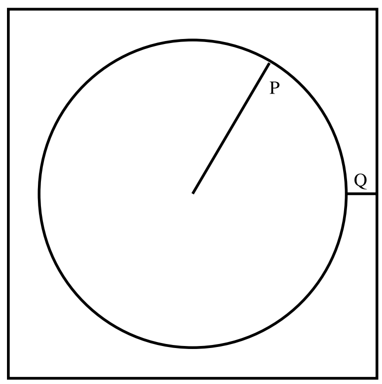

Figure 1.1: Top view of the canonical roulette

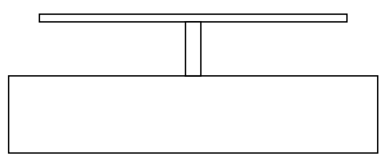

Figure 1.2: Side view of the canonical roulette

There are two reference points P and Q along the edge of the wheel. The former is on the wheel itself and the latter is on the table. (see figures 1.1 and 1.2).

A spin of the wheel produces a number ω greater than or equal to zero and less than one by the following procedure. After the wheel comes to a stop we measure the counterclockwise arc QP (see figure 1.1) on the circumference of the wheel. The value of ω is simply the length of the arc.

The number ω can also be expressed in terms of an angle. Indeed, if 0 denotes the center of the wheel and α denotes the angle QOP measured in radians and in the counterclockwise direction, then

$$\omega = \frac{\alpha}{2\pi}$$

We need to imagine that the operation of this device is completely automated so that the only human intervention consists of pushing a trigger-type button which imparts the wheel its initial spin.

We may assume for instance that while we hold down the trigger a magnetic field goes into action which imparts the wheel a spin whose final magnitude is proportional to the time we hold the trigger depressed. After that, by inertia, the wheel undergoes a very large number of revolutions and when it finally comes to a stop, automatically somehow, out of the device comes ω.

To be specific we shall further assume that ω appears in the form of a voltage at some electrical terminal of the device.

This whole package will be referred to here and in the following as the *canonical roulette.*

Remarks: Although this will not happen very often, if P and Q should happen to fall on top of each other, we shall agree to set $\omega = 0$ in this case.

2 Fortune Wheels

The canonical roulette is a physical device that may be used to output a random number on the continuous range between 0 and 1. The canonical roulette device may be modified to output one of a finite set of values at random.

First of all, we remove the point P and instead apply on the edge of the wheel, two pieces of colored tape, say red and green. We recall that the wheel was supposed to have a circumference of unit length (one meter), thus if no portion of the edge is to remain uncovered, the total amount of tape used should measure one meter. In other words, if we use a total of p_1 and p_2 of a meter respectively of red and green tape, we should have $p_1 + p_2 = 1$.

Finally, at the point Q (which is on the table) we now fasten a little arrow that is pointing toward the center of the wheel. (see figure 2.3).

We give the wheel a spin and wait until it stops. If the arrow points towards the red tape we say that the outcome of the spin is *red* and if the arrow points toward the green tape then we call the outcome of the spin *green.*

For example in Figure 1.3 below, we have represented a wheel with $p_1 = 3/4$ and $p_2 = 1/4$. If we spin this wheel 1000 times, roughly three-fourths of the time it would come up *red* and the remaining one-fourth of the time it would have an outcome of *green.*

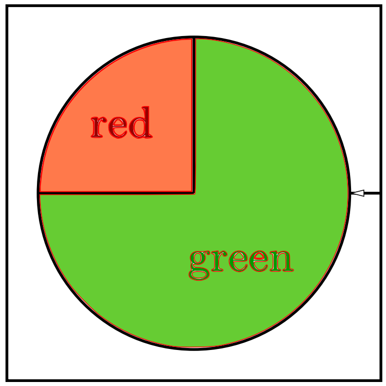

Figure 2.3: A fortune wheel with 3/4 and 1/4

Of course we need not restrict ourselves to using only two kinds of tapes. Indeed, we could just as well cover the edge of the wheel with n different kinds of tapes labeled by $1, 2, \ldots, n$ of respective lengths $p_1, p_2, \ldots, p_n$, producing a wheel with outcome i being selected with probability p_i.

3 The Game of Craps

How to Play Craps:

Craps is a dice game that is played at most gambling houses. There are several different variations of it and several ways of betting, thus, to avoid misunderstandings, we shall spell out here in detail a simple version of the game.

There is one player who is called the "shooter" who rolls the dice. He bets against the "house" that he will win the outcome of a round of craps. There are other 'customers' who are allowed to place bets on the outcome of rolls of the dice and of the round of the game.

To play one round of craps, the shooter casts a pair of dice.

a) If the roll is a 7 or 11, the shooter wins the round.
b) If the roll is 2,3 or 12, the shooter loses the round.
c) If the first cast is a 4,5,6,8,9 or 10 the the shooter keeps casting the dice until:
    1) a 7 occurs
    2) the number from the first cast occurs again

In the former case the shooter loses, in the latter case the shooter wins. If the shooter had bet $1 on the outcome of the round, the house takes the dollar if the shooter loses, and gives a dollar if the shooter wins.

We may now make our description a little more mathematical and construct an experiment based on the activity of the customer during one round of the game. Note that when the shooter plays a single round of craps, he often produces a large quantity of random numbers, the most relevant ones are the three quantities:

$U =$ the result of the first cast

$V =$ the result of the cast that decides the round

$$X = \begin{cases} 1 & \text{if the shooter wins} \\ 0 & \text{if the shooter loses} \end{cases}$$

We shall let E denote the experiment consisting in carrying out the activity of the customer recording the vector (U, V, X) where U,V and X are defined above.

We can see that if $U = 2, 3, 7, 11$, or 12 then the round is decided on the first cast, $U = V$, and $X = \begin{cases} 1 & \text{if } U = 7, 11 \\ 0 & \text{if } U = 2, 3, 12 \end{cases}$.

If $U = 4, 5, 6, 8, 9, 10$ then the game proceeds according to the following flow chart.

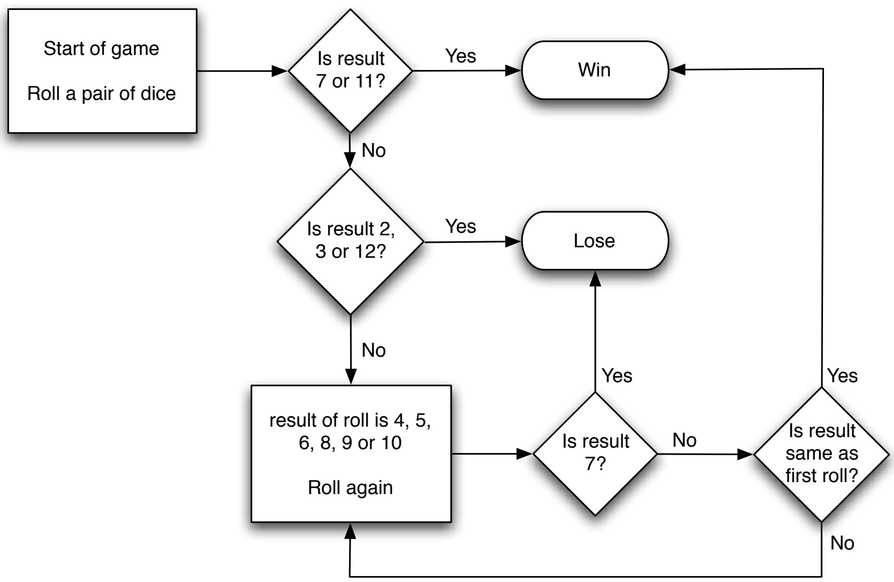

Figure 3.4: A flow chart to decide what happens after the first roll.

We can then conclude that the set Ω of all possible outcomes of E consists of the following 17 vectors:

$$(2, 2, 0), (3, 3, 0), (4, 4, 1), (4, 7, 0), (5, 5, 1), (5, 7, 0)$$
$$(6, 6, 1), (6, 7, 0), (7, 7, 1), (8, 7, 0), (8, 8, 1), (9, 7, 0)$$
$$(9, 9, 1), (10, 7, 0), (10, 10, 1), (11, 11, 1), (12, 12, 0)$$

To illustrate the terminology, the sentence '*the customer wins*' characterizes the event consisting of all those triplets for which the third component is 1, namely the 8 outcomes

$$(4, 4, 1), (5, 5, 1), (6, 6, 1), (7, 7, 1)$$
$$(8, 8, 1), (9, 9, 1), (10, 10, 1), (11, 11, 1)$$

The goal here will be to create a fortune wheel such that spinning it once represents playing one round of craps. Since there are 17 possible outcomes of this game, we will represent one round of craps by a wheel with 17 regions on it. We shall build this wheel in steps.

Our first step will be to produce a wheel that simulates $U =$ the first cast of the dice. It is clear that casting a pair of dice and recording the sum is equivalent to operating the following random device.

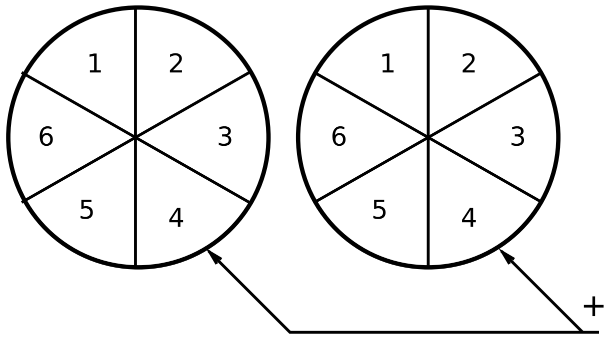

Figure 3.5: A device to cast two dice and record their sum

Our goal is to produce U on a single roulette spin. Since no matter what A turns out to be, B is obtained by spinning a roulette with 6 equal spots, all we have to do (if we want a single wheel) is to reproduce along side each of the arcs of the first roulette an exact replica of the second roulette. The resulting fortune wheel is depicted in figure 3.6 below.

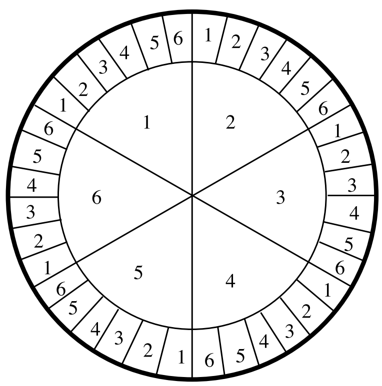

Figure 3.6: Two dice represented on one wheel

The resulting fortune wheel has 36 equally spaced arcs, each labeled with a pair (i, j) where i represents the value of the first die roll and j represents the value of the second die roll. Next, we write along each of the arcs the value of $i + j$ to obtain figure 3.7 below.

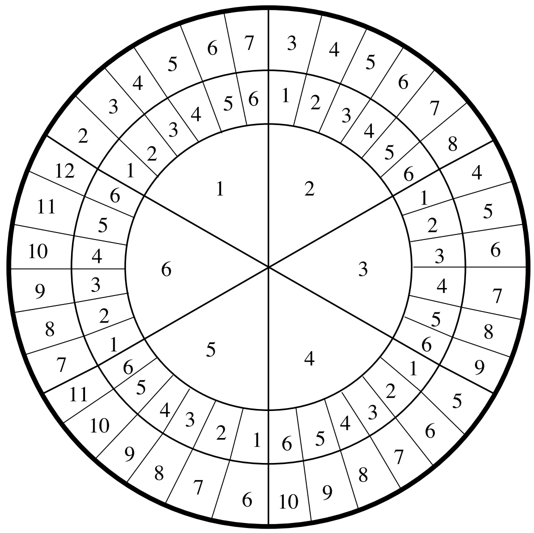

Figure 3.7: Two dice and their sum

We may now cover each of the 36 arcs using only 11 different tapes (representing the numbers 2 though 12). We will cover the part where $i+j=2$ with tape #2, the part where $i+j=3$ with tape #3, etc.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 3.8: Addition table for the numbers 1-6

There are 36 arcs of equal length, so each is 1/36 of the total length of the circumference. By looking at the table in figure 3.8 or by simply counting the number of arcs that are labeled with each number we conclude that since the total circumference of the circle is 1 meter we need a total of

1/36 of a meter for each of tape # 2 and # 12
$2/36 = 1/18$ of a meter for each of tape # 3 and # 11
$3/36 = 1/12$ of a meter for each of tape # 4 and # 10
$4/36 = 1/9$ of a meter for each of tape # 5 and # 9
5/36 of a meter for each of tape # 6 and # 8
$6/36 = 1/6$ of a meter of tape # 7

To simulate U, we can just spin the wheel just produced and record the number of the tape that stops in front of the arrow.

It is clear that we can simulate U just as well by any wheel that can be obtained by rearranging the tape in any order we wish. In particular, we can lump together in successive bunches all of the regions with the same type of tape. Indeed it is clear that what matters is not how each kind of tape is cut or in what order these various pieces are applied, but how much tape of each particular kind is used in building the wheel.

This observation allows us to conclude that casting a pair of dice is and recording the sum is equivalent to spinning the wheel shown in figure 3.9 below.

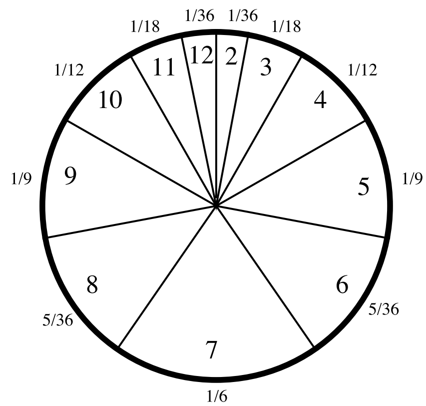

Figure 3.9: A wheel that represents throwing a pair of dice and recording the sum only

Our next task is to modify the wheel shown in figure 3.9 so that it not only produces U, but also V and X as well. We note that when $U = 2, 3, 7, 11$, or 12 that the values of V and X are completely determined already.

If U is one of the remaining possibilities then there are two choices for the value of the pair (V, X). Consider the case when $U = 4$, then V is determined by spinning the wheel in figure 3.9 until either a 4 or a 7 comes up. The arcs labeled by 4 and 7 are 'live' and all of the remaining are 'dead.' One third of the 'live' arc is labeled by a 4 and two thirds of the 'live' arc is labeled by a 7 because

$$P[V = 4|V = 4 \textit{ or } 7] = \frac{P[V = 4 \;\&\; V = 4 \textit{ or } 7]}{P[V = 4 \textit{ or } 7]} = \frac{1/12}{1/12 + 1/6} = 1/3$$

and similarly

$$P[V = 7|V = 4 \textit{ or } 7] = \frac{P[V = 7 \;\&\; V = 4 \textit{ or } 7]}{P[V = 4 \textit{ or } 7]} = \frac{1/6}{1/12 + 1/6} = 2/3.$$

In other words, when $U = 4$, V can just as well be determined by spinning the 'healthy' wheel shown in figure 3.10 below.

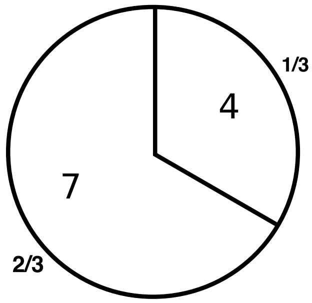

Figure 3.10: A wheel representing the remainder of a round once it is determined that $U = 4$.

Similarly, when $U = 5$, V is either 5 or 7. Crippling the wheel so that only these two numbers come up means that V can be determined by spinning the healthy wheel shown in figure 3.11 below.

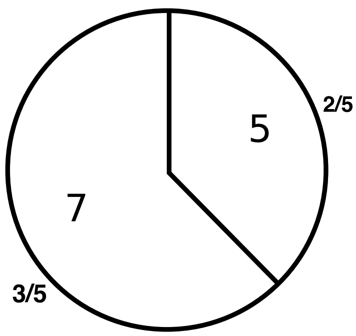

Figure 3.11: A wheel representing the remainder of a round once it is determined that $U = 5$.

The lengths of the arcs on the healthy wheels are computed from the conditional probabilities $P[V = 5|V = 5 \text{ } or \text{ } 7]$ and $P[V = 7|V = 5 \text{ } or \text{ } 7]$ which are easily calculated as $2/5$ and $3/5$ respectively.

And when $U = 6$ we can get V by spinning the wheel below where the lengths of the arcs labeled 6 and 7 are determined from $P[V = 6|V = 6 \text{ } or \text{ } 7]$ and $P[V = 7|V = 6 \text{ } or \text{ } 7]$.

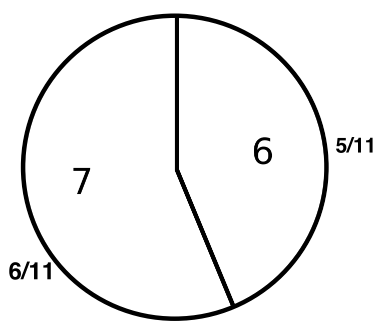

Figure 3.12: A wheel representing the remainder of a round once it is determined that $U = 6$.

The wheels needed for the cases when $U = 8, 9, 10$ can be obtained by changing $4, 5,$ and 6 in the wheels above into $10, 9$ and 8 respectively.

We are now ready to construct our desired fortune wheel. We need now only reproduce along each side of the arc in figure 3.9 a replica of the wheel that is used to obtain V. In other words along the arc labeled by 4 in figure 3.9 we reproduce the wheel of figure 3.10, along the arc labeled by 5 we use the wheel of figure 3.11, etc. This produces the wheel in figure 3.13 below.

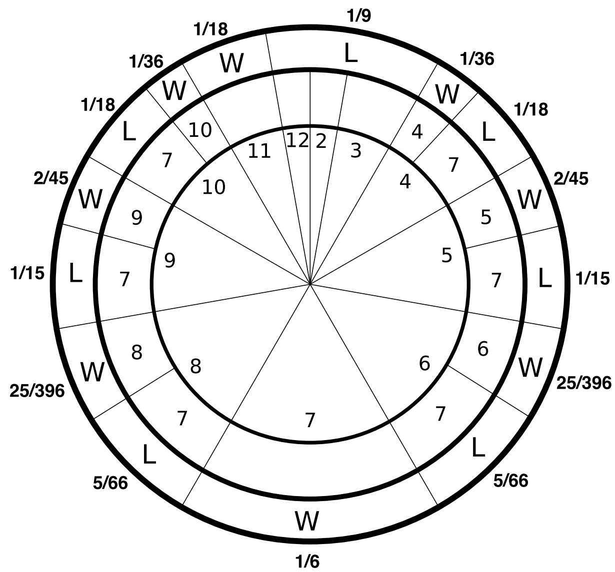

Figure 3.13: A wheel representing a single round of craps

We can read from this wheel the probability of winning a single round of craps. The shooter wins when the wheel points to a region that is labeled by a W in the outer ring, and loses when it points to a L in the outer region. To find the probability that the shooter wins means that we should add up the lengths of the arcs that are labeled with a W.

$$\frac{1}{18} + 2\frac{1}{3}\frac{1}{12} + 2\frac{2}{5}\frac{1}{9} + 2\frac{5}{11}\frac{5}{36} + \frac{1}{6} = \frac{244}{495} = .4929$$

What this tells us is that the shooter wins this game surprisingly often. As in any casino game the odds favor the house, but the probability that the house wins a single game is just slightly over 50%. In most other casino games the house is favored much more.

The expected value of a bet will be the amount of money that the better expects to win (or lose) on 'average.' In a casino game of a customer betting against the house, the expected value will almost always be negative (how else can the house continue to make money by hosting games?). For some games the expected value of a bet favors the house only slightly, for other games the house is very strongly favored.

Say that the customer loses \$$B$ on a game if he loses, and wins \$$A$ if the outcome of the game is in his favor. If the probability of winning the game is P_{WIN} and the probability of losing the game is P_{LOSE} then the expected value of the game is

$$A \cdot P_{WIN} - B \cdot P_{LOSE}$$

A \$1 bet on the come line of the table will pay off \$1 if the shooter wins. The expected value of the outcome this bet will be simply $1 \cdot .4929 - 1 \cdot .5071 = -.0142$. That means that on average the player will lose 1.4 cents per dollar that is bet.

More betting:

At the end of this section we describe a number of other bets that are made at the craps table. The probability of winning those games may also be read off of this wheel with little difficulty.

A customer may place an 'any seven' bet that the next roll will be a 7. For this bet we look at the inner part of the wheel only and see that the probability that the next roll is a seven is 1/6. The customer will win this bet only one in six times.

The payoff for an 'any seven' bet is four times the bet made, so the payoff for a $1 dollar bet will be $4. The expected value of this bet is $4 \cdot \frac{1}{6} - 1 \cdot \frac{5}{6} = -.1667$ which means that on average the better will lose 16 cents for every dollar bet. This bet is much more advantageous to the house than the come line bet where the expected value was still in favor of the house but value is less than one-tenth of the any seven bet.

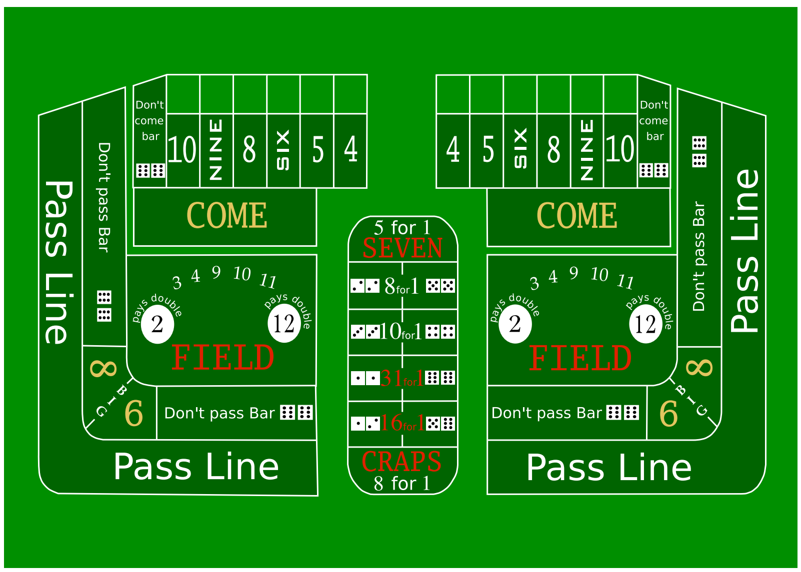

Figure 3.14: a craps betting table

Betting Terminology:

any craps

a bet that the next roll will be 2, 3, or 12. This bet pays 7:1.

any seven

a bet that the next roll will be 7. This bet pays 4:1.

big 6

a bet that a 6 will be rolled before a 7 comes up. This bet pays even money. A place bet on 6 pays 7:6 but is identical otherwise. The place bet is preferred since it has a better payoff than the big 6 bet.

big 8

a bet that an 8 will be rolled before a 7 comes up. This bet pays even money. A place bet on 8 pays 7:6 but is identical to the big 8 bet otherwise. The place bet is preferred because of the better payoff.

pass bet

a bet that the dice will pass, also known as a 'pass line' bet. This bet is generally placed immediately before a 'come out' roll, although you can make or increase this bet at any time.

don't pass bet

a bet that the dice will not pass. This bet can be placed only immediately before a 'come out' roll. One result (either the 2 or the 12, depending on the casino) will result in a push. A don't pass bet can be taken down, but not increased, after the come-out roll.

field bet

a bet that the next roll will be 2, 3, 4, 9, 10, 11, or 12. This bet pays even money for 3, 4, 9, 10, and 11, and usually pays 2:1 for 2 or 12. Some casinos pay 3:1 for either the 2 or 12 (but not both), and some casinos may make the 5 instead of the 9 a field roll.

hard way

a bet on 4, 6, 8, or 10 that wins only if both dice show the same face; e.g., 'hard 8' occurs when each die shows a four. The 'hard 8' bet wins if the dice come up double 4 before a 7 or any other 8.

point

if a 4, 5, 6, 8, 9, or 10 is rolled on the come out roll, then this number becomes the 'point.' The shooter must roll the point again, before rolling a seven, in order for the dice to 'pass.' A 'come point' is just the number that is serving as a point for a come bet.

shooter

the player who is rolling the dice. The shooter must place a 'line' bet ('pass' or 'don't pass') in order to be eligible to roll the dice. Of course, the shooter can place other bets in addition to the required 'line' bet. Most shooters (and players) tend to play the 'pass' line. Note that shooters who make 'don't pass' bets are not betting against themselves, they are simply betting that the dice will not 'pass.'

EXERCISE:

The table shown on the next page lists a number of common craps bets and their description. For each bet, calculate the probability of winning, the probability of losing, and the house advantage (= the negative of the expected value of the bet expressed as a percentage of the original bet).

Name of Bet	Description	Payoff per $1	P(Win)	P(Lose)	House Advantange
Pass Bet	2,3,12 -lose 7,11-win 4,5,6,8,9,10- this is the point shooter rolls again until either 7 or the point comes up if point is first then win if 7 is first then lose	$1	.492929	.507070	1.4%
Don't Pass Bet	2,3- win 12-roll again 7,11- lose 4,5,6,7,9,10- this is the point shooter rolls again until either 7 or the point comes up if the point is first then lose if 7 is first then win	$1			
Field Bet	The next roll is 2,3,4,9,10,11,12-win 5, 6, 7, 8-lose	$2 for 2 or 12 $1 otherwise			
Any Craps	The next roll is 2, 3, 12-win 4, 5, 6, 7, 8, 9, 10, 11- lose	$7			
Any 7	The next roll is a 7- win 2,3,4,5,6,8,9,10,11,12- lose	$4			
Big 6	If a 6 is rolled before a 7-win If a 7 is rolled before a 6-lose	$1			
Big 8	If a 8 is rolled before a 7-win If a 7 is rolled before a 8-lose	$1			
4 Hardway	If a pair of twos is rolled before a 7 or before a 1 and 3- win otherwise lose	$7			
10 Hardway	If a pair of fives is rolled before a 7 or before a 4 and a 6- win otherwise lose	$7			
6 Hardway	If a pair of threes is rolled before a 7 or before 2&4 or 1&5-win otherwise lose	$9			
8 Hardway	If a pair of fours is rolled before a 7 or before 2&6 or 3&5-win otherwise lose	$9			