# A BASIC INEQUALITY

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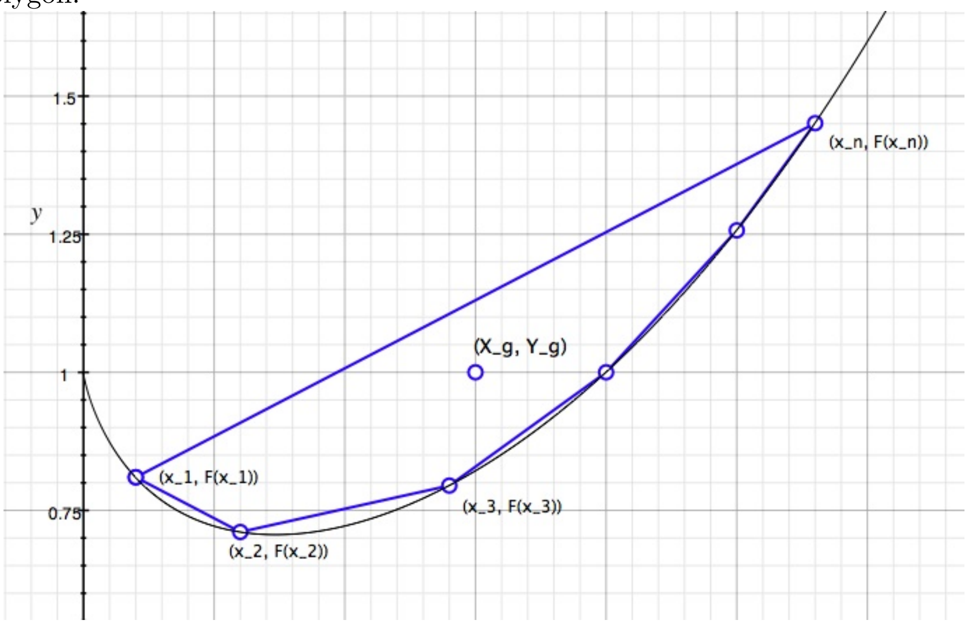
There is an inequality that we will use repeatedly in this class that follows from a single picture.

Assume that we have points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ which lie on a curve which is convex up. If our curve is $y = F(x)$ then the points on the curve will be $(x_1, F(x_1)), (x_2, F(x_2)), \ldots, (x_n, F(x_n))$. Also assume that we have weights $m_1, m_2, \ldots, m_n$ with $m_i \geq 0$ such that $\sum_{i=1}^{n} m_i = 1$. In the picture below I have drawn these points on an example of such a curve. I connected the points together to form a shape and because I have assumed that $y = F(x)$ is convex up then the connected points forms a convex polygon.



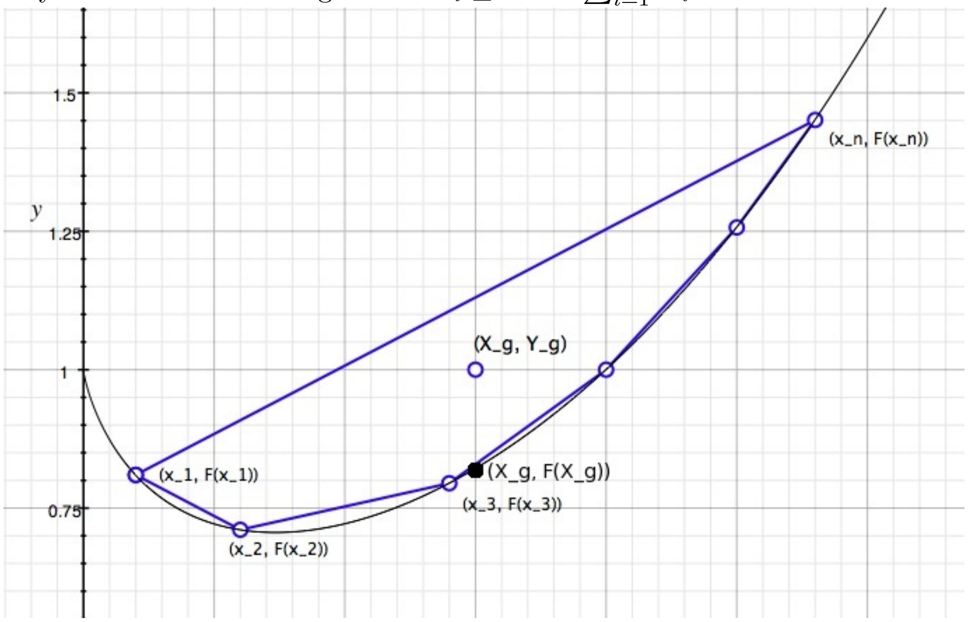

Now let $(X_g, Y_g)$ be the weighted average of all of these points with respect to the weights $m_i$ (the center of gravity), that is

$$X_g = m_1x_1 + m_2x_2 + \cdots + m_nx_n = \sum_{i=1}^{n} m_ix_i$$

and

$$Y_g = m_1F(x_1) + m_2F(x_2) + \cdots + m_nF(x_n) = \sum_{i=1}^{n} m_iF(x_i) \ .$$

I have drawn approximately where $(X_g, Y_g)$ will be if all of the $m_i = 1/n$ and this point represented the true center of gravity, but it is possible to choose weights in such a way that $(X_g, Y_g)$ is almost anywhere within the convex region connecting the points. The important thing to note is that the point $(X_g, Y_g)$ will always lie within the polygon traced out by the blue lines as long as the $m_i \geq 0$ and $\sum_{i=1}^n m_i = 1$.




Since the point $(X_g, Y_g)$ is in the interior of the region and the point $(X_g, F(X_g))$ is on the curve, we can conclude that $(X_g, Y_g)$ is above $(X_g, F(X_g))$ and hence

$$Y_g \geq F(X_g)$$

that is, we know that

$$Y_g = \sum_{i=1}^{n} m_i F(x_i) \geq F\left(\sum_{i=1}^{n} m_i x_i\right) = F(X_g) \ .$$

Note that there is an analogous that you can derive if $F(x)$ happens to be concave down. Alternatively, if you have an $F(x)$ which is concave down, then $-F(x)$ is concave up.

We are going to use this result now to show another inequality involving two sets of probabilities that sum to 1. Take $p_1, p_2, \ldots, p_n$ as one set and $q_1, q_2, \ldots, q_n$ such that $q_i \neq 0$. Now we apply the inequality that we just derived with $x_i = p_i/q_i$ and $m_i = q_i$. We will also take $F(x) = x \log_b(x)$. Notice that since

$$F''(x) = \frac{1}{\ln(b)x}$$

then $F(x)$ is concave up for $x > 0$.

For our $X_g$ we obtain

$$X_g = \sum_{i=1}^{n} m_i x_i = \sum_{i=1}^{n} q_i(p_i/q_i) = \sum_{i=1}^{n} p_i = 1$$

and so $F(X_g) = 1\log_b(1) = 0$.

For our $Y_g$ we have

$$Y_g = \sum_{i=1}^{n} m_i x_i \log_b(x_i) = \sum_{i=1}^{n} q_i(p_i/q_i)\log_b(p_i/q_i) = \sum_{i=1}^{n} p_i(\log_b(p_i) - \log_b(q_i)) \ .$$

Our inequality says that

$$\sum_{i=1}^{n} p_i(\log_b(p_i) - \log_b(q_i)) \geq 0$$

or, by rearranging terms,

$$\sum_{i=1}^{n} p_i \log_b(p_i) \geq \sum_{i=1}^{n} p_i \log_b(q_i) \ .$$

**Exercises:**

(1) For $n > 0$, take values $a_i > 0$ and probabilities $p_i$ such that $\sum_{i=1}^{n} p_i = 1$. Show one of the two inequalities holds (hint: $exp(x)$ is a concave up function).

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \qquad \leq \text{ or } \geq \qquad p_1 a_1 + p_2 a_2 + \cdots + p_n a_n.$$

(2) Use the convex/concave identity discussed in class to prove for all $a, b > 0$,

$$\frac{1}{\sqrt[6]{2}}\sqrt{a^2 + b^2} \leq \sqrt[3]{a^3 + b^3} \ .$$

Show precisely how you use the identity.

(3) Using the convex inequality, prove for $n \geq 1$ that

$$\sum_{i=0}^{n-1} \sqrt{3i^2 + 3i + 1} \leq n^2 \ .$$

Hint: if $f(x) = \sqrt{x}$, then $f''(x) = -1/4x^{-3/2}$. Also $3a^2 + 3a + 1 = (a+1)^3 - a^3$.

(4) Use either the convex or concave inequality to derive an inequality of the form

$$\frac{1}{n}(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}) \leq c\sqrt{n+1}$$

for some appropriate value of $c$ which does not depend on $n$. That is, choose some values $x_1, x_2, \ldots, x_k$ and weights $m_1, m_2, \ldots, m_k$ where $\sum_i m_i = 1$ and an appropriate function $f(x)$ (state clearly if your function is concave up or down) to see how the left and right hand side of the expressions above compare. You may need the identity that $\sum_{i=1}^{n} i = n(n+1)/2$.