FINAL EXAM (PART I) OF MATH 5020

ASSIGNED: DECEMBER 1, 2003 - DUE: JANUARY 5, 2003

There are a few instructions for this exam that I want to give in advance:

- The purpose of this exam is to encourage you to write clearly and succinctly. For that reason I am not asking you to give me answers, I am only looking for solutions. Each of the following questions that I am asking you has the answer given. This in this exam you must explain that answer.
- Expect to write and rewrite your solution to each problem 2 to 3 times. In order for you to receive full credit for your answer your explanation must be complete and crystal clear.
- The 'division principle' is difficult to explain clearly. You are hereby warned that any solution that you give using it will be marked down unless it is applied clearly and correctly.
- I want you to work alone on this exam. Once you have read this consider yourself under 'exam conditions' and do not discuss the exam with anyone else. You may ask me questions although I would like you to ask them in the FORUM, this way everyone has the chance to read the same instructions/information.
- There are nine questions on this exam but you are only required to answer five. There are choices that you can make about which questions to answer so please read the instructions carefully. I ask that you do any two of the first 3 questions, any two of questions 4, 5 and 6, and any one question from 7, 8, or 9.

(1) Ontario has a lottery called LOTTO 6/49, the game is played by choosing six numbers from 1 to 49. Six winning numbers and a bonus are chosen by a machine and a player wins by matching some or all of the winning numbers.
  (a) What is the probability of matching all 6 number correctly?
  Answer: The probability of matching all 6 numbers is 1/13983816.
  (b) What is the probability of matching 5 numbers out of 6 and the bonus number?
  Answer: The probability of matching 5 out of the 6 winning numbers and the bonus is 1/2330636.
  (c) What is the probability of matching 4 numbers out of 6?
  Answer: The probability of matching 4 out of 6 numbers is $645/665896 \approx 1/1033$.

  For further information about LOTTO 6/49 including the odds given in this problem, see `http://lotteries.olgc.ca/consumer_hp.jsp` (then click on the 6/49 icon)

(2) Use basic counting techniques to answer the following questions.
  (a) How many ways are there of coloring n different objects using k colors?
  Answer: The number of ways of coloring n different objects using k colors is k^n.
  (b) How many ways are there of coloring n different objects using k colors such that each color is used at most once?
  Answer: The number of ways of coloring n different objects using k colors such that each color is used at most once is equal to $k(k-1)\cdots(k-n+1)$.

(c) How many ways are there of coloring n identical objects using k colors?
Answer: The number of ways of coloring n identical objects using k colors is $\binom{n+k-1}{n}$.

(d) How many ways are there of coloring n identical objects using k colors such that each color is used at most once?
Answer: The number of ways of coloring n identical objects using k colors such that each color is used at most once is equal to $\binom{k}{n}$.

(3) Use basic counting techniques to answer the following questions.

(a) How many distinct ways are there of rearranging the letters of the word GREATGRANDFATHER?
Answer: The number of ways of rearranging the letters of the word GREATGRANDFATHER is 72,648,576,000.

(b) How many distinct ways are there of rearranging the letters of the word GREATGRANDFATHER such that the vowels are not consecutive.
Answer: The number of ways of rearranging the 16 letters of GREATGRANDFATHER such that the vowels do not appear consecutively is 13,172,544,000.

(c) How many ways are there of rearranging the letters of the word GREATGRANDFATHER such that the sequence of letters 'GRAND' appears consecutively somewhere in the word (in that order)?
Answer: The number of ways of rearranging the letters of the word GREATGRANDFATHER such that the sequence of letters 'GRAND' appears consecutively is 29,937,600.

(4) Find all integers n such that $\phi(n) = 12$.
Answer: Only $n = 13, 21, 26, 28, 36$ or 42 have $\phi(n) = 12$.

(5) Prove that if $r_1, r_2, \ldots, r_{\phi(m)}$ is a reduced residue system modulo m, and m is odd, then $r_1 + r_2 + \cdots + r_{\phi(m)} \equiv 0 \pmod{m}$. Hint: You can explain why this must be true by showing that $2r_1, 2r_2, \ldots, 2r_{\phi(m)}$ is also a reduced residue system and then explaining why the sum of any reduced residue system must be equivalent to the sum of any other reduced residue system modulo m.

(6) Demonstrate the following identity by counting a set of objects in two different ways:

$$\binom{n}{4} + \binom{n-1}{3} + \binom{n-1}{2} = \binom{n+1}{4}$$

(7) Let $D_n = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_n}\right)$. Prove by induction that

$$D_n = \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ S = \{s_1, s_2, \ldots, s_{|S|}\}}} \frac{(-1)^{|S|}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}}$$

Example: $D_1 = \left(1 - \frac{1}{p_1}\right)$ and if $S \subseteq \{1\}$ then $S = \{\}$ or $S = \{1\}$. Therefore $D_1 = 1 + \frac{-1}{p_1}$.
$D_2 = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)$ and if $S \subseteq \{1, 2\}$ then S is one of $\{\}, \{1\}, \{2\}, \{1, 2\}$. Therefore $D_2 = 1 + \frac{-1}{p_1} + \frac{-1}{p_2} + \frac{1}{p_1 p_2}$.
$D_3 = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)$ and if $S \subseteq \{1, 2, 3\}$ then S is one of $\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$. Therefore $D_3 = 1 + \frac{-1}{p_1} + \frac{-1}{p_2} + \frac{-1}{p_3} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \frac{1}{p_2 p_3} + \frac{-1}{p_1 p_2 p_3}$.

(8) Let $p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k})$ with $p_{n,1} = 1$ and $p_{n,k} = 0$ if $k > n$. We give a table of $p_{n,k}$ for $n, k \le 5$.

n/k	1	2	3	4	5	$\cdots$
1	1	0	0	0	0	
2	1	2	0	0	0	
3	1	6	6	0	0	
4	1	14	36	24	0	
5	1	30	150	240	120	
$\vdots$						

Show by induction that

$$p_{n,k} = k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots + (-1)^{k-1}\binom{k}{k-1}1^n$$

(9) Let $p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k})$ with the initial conditions $p_{n,1} = 1$ and $p_{n,k} = 0$ for $k > n$. Show by induction on n that $p_{n,k}$ is equal to the number of ways of putting n different items in k different boxes such that there is at least one thing in each box.