FINAL EXAM (PART I) OF MATH 5020

ASSIGNED: DECEMBER 1, 2003 - DUE: JANUARY 5, 2003

There are a few instructions for this exam that I want to give in advance:

- The purpose of this exam is to encourage you to write clearly and succinctly. For that reason I am not asking you to give me answers, I am only looking for solutions. Each of the following questions that I am asking you has the answer given. This in this exam you must explain that answer.
- Expect to write and rewrite your solution to each problem 2 to 3 times. In order for you to receive full credit for your answer your explanation must be complete and crystal clear.
- The 'division principle' is difficult to explain clearly. You are hereby warned that any solution that you give using it will be marked down unless it is applied clearly and correctly.
- I want you to work alone on this exam. Once you have read this consider yourself under 'exam conditions' and do not discuss the exam with anyone else. You may ask me questions although I would like you to ask them in the FORUM, this way everyone has the chance to read the same instructions/information.
- There are nine questions on this exam but you are only required to answer five. There are choices that you can make about which questions to answer so please read the instructions carefully. I ask that you do any two of the first 3 questions, any two of questions 4, 5 and 6, and any one question from 7, 8, or 9.

(1) Ontario has a lottery called LOTTO 6/49, the game is played by choosing six numbers from 1 to 49. Six winning numbers and a bonus are chosen by a machine and a player wins by matching some or all of the winning numbers.

(a) What is the probability of matching all 6 number correctly?
Answer: The probability of matching all 6 numbers is 1/13983816.
Solution: The total number of possible tickets in this game is equal to the number of ways of choosing 6 items from a set of size 49 and this we know is equal to $\binom{49}{6}$. There is exactly one winning ticket which matches all 6 of the winning numbers and since the probability of matching all 6 numbers is equal to (the number of tickets which match all 6 correctly)/(the total number of possible tickets) then this is equal to $1/\binom{49}{6}$.

(b) What is the probability of matching 5 numbers out of 6 and the bonus number?
Answer: The probability of matching 5 out of the 6 winning numbers and the bonus is 1/2330636.
Solution: There are $\binom{6}{5}$ ways of picking 5 of the winning 6 numbers to go on a ticket and the bonus will be the last number. Therefore there are $\binom{6}{5}$ tickets which match 5 of the 6 winning numbers and the bonus. The probability of matching 5 out of the 6 winning numbers and the bonus = (the number of tickets which match 5 of the the 6 numbers and the bonus)/(the total number of tickets) = $\binom{6}{5}/\binom{49}{6}$.

(c) What is the probability of matching 4 numbers out of 6?
Answer: The probability of matching 4 out of 6 numbers is $645/665896 \approx 1/1033$.
Solution: A ticket which has 4 of the 6 winning numbers will also have 2 of the 43

numbers which are non-winning. There are $\binom{6}{4}$ ways of picking 4 of the 6 winning numbers to go on a ticket and $\binom{43}{2}$ ways of choosing 2 numbers of the remaining 43 non-winning numbers. Therefore by the multiplication principle the number of tickets which match exactly 4 of the 6 winning numbers is $\binom{6}{4}\binom{43}{2}$. The probability of matching 4 out of 6 numbers = (the number of tickets which match exactly 4 of the 6 winning numbers)/(the total number of tickets) = $\binom{6}{4}\binom{43}{2}/\binom{49}{6}$.

For further information about LOTTO 6/49 including the odds given in this problem, see `http://lotteries.olgc.ca/consumer_hp_games.jsp?game=649&flag=null`

(2) Use basic counting techniques to answer the following questions.

(a) How many ways are there of coloring n different objects using k colors?
Answer: The number of ways of coloring n different objects using k colors is k^n.
Solution: Each object can be colored with one of k different colors and this can be done in k different ways. Consider a procedure which colors the first object, then the second, and then the third, ..., and finally colors the n^{th} object. The number of outcomes of this procedure is the number of ways of coloring the n different objects with k different colors and by the multiplication principle this is equal to k^n.

(b) How many ways are there of coloring n different objects using k colors such that each color is used at most once?
Answer: The number of ways of coloring n different objects using k colors such that each color is used at most once is equal to $k(k-1)\cdots(k-n+1)$.
Solution: Consider a procedure which colors the first object with one of k different colors, the second object can be colored with any of the $k-1$ colors which were not used, the third item with any of the $k-2$ colors which were not used to color the first two items, etc. The n^{th} object can be colored with any of the $k-n+1$ colors which were not used to color the first $n-1$ objects. The number of outcomes of this procedure is equal to the number of ways of coloring the n items with k different colors so that no color is used more than once and by the multiplication principle this will be equal to $k(k-1)\cdots(k-n+1)$.

(c) How many ways are there of coloring n identical objects using k colors?
Answer: The number of ways of coloring n identical objects using k colors is $\binom{n+k-1}{n}$.
Solution: For every coloring of n identical objects with k different colors we may take the objects that were colored with the first color and put them in the first of k different bins, the objects colored with the second color and put them in the second bin, etc. In the end we can erase the colors on these objects so that they are all identical because we know that the ones in the first bin were colored with the first color and the ones in the second bin were colored with the second color, etc. Therefore the number of colorings of n identical objects with k different colors is equal to the number of ways of placing n identical items in k different bins. From our discussion in class we know that the number of ways of placing n identical objects in k different bins is equal to $\binom{n+k-1}{n}$.

(d) How many ways are there of coloring n identical objects using k colors such that each color is used at most once?
Answer: The number of ways of coloring n identical objects using k colors such that each color is used at most once is equal to $\binom{k}{n}$.
Solution: If $n \leq k$ then choose n of the k colors and color the n identical objects using these n colors that were chosen so that each color is used exactly once. The outcomes of this procedure is equal to the number of ways of coloring the n different items each with a different color and is also equal to $\binom{k}{n}$ because this is the number of ways of choosing n colors from a set of k colors. If $n > k$ then the number of ways of coloring

n identical items with k different colors with each color used at most once is 0 because there will not be enough colors to color each object a different color and we also know that $\binom{k}{n}$ is also equal to 0.

(3) Use basic counting techniques to answer the following questions.

(a) How many distinct ways are there of rearranging the letters of the word GREAT-GRANDFATHER?

Answer: The number of ways of rearranging the letters of the word GREATGRAND-FATHER is 72,648,576,000.

Solution: The word "GREATGRANDFATHER" has 2 G's, 3 R's, 2 E's, 3 A's, 2 T's, and one each of N, D, F, H (16 letters in total). Imagine an ordered list of 16 positions that we will fill with letters. First we will choose 2 of the 16 positions that will contain the letter G, next we will choose 3 of the remaining 14 positions that will contain the letter R, then we will choose 2 of the remaining 11 positions that contain the letter E, then we will choose 3 of the remaining 9 positions to contain the letter A, then 2 of the remaining 6 positions that will contain the letter T, then 1 of the remaining 4 positions for the N, then 1 of the remaining 3 positions for the D, then 1 of the remaining 2 positions for the F and the H will be in the remaining position. The number of outcomes of this procedure is equal to the number of ways of rearranging the letters of the word GREATGRANDFATHER and is also equal to $\binom{16}{2}\binom{14}{3}\binom{11}{2}\binom{9}{3}\binom{6}{2}\cdot 4\cdot 3\cdot 2$ by the multiplication principle.

(b) How many distinct ways are there of rearranging the letters of the word GREAT-GRANDFATHER such that the vowels are not consecutive.

Answer: The number of ways of rearranging the 16 letters of GREATGRANDFATHER such that the vowels do not appear consecutively is 13,172,544,000.

Solution: The word GREATGRANDFATHER has 5 vowels and 16 letters in total. Imagine an ordered list of 12 positions that we will fill with letters (plus 4 blank positions which we will withhold to insert in one of the steps of our procedure). We will choose 5 of the 12 positions to contain vowels and then insert 4 new positions immediately following each of the vowels to ensure that they are each separated by at least one space. Of the 5 spaces that we have now reserved for vowels choose 2 of them to contain E's, the other 3 will be filled with A's. The remaining 11 spaces may now be filled with the consonants. First choose 2 of the 11 remaining spaces that will contain G's, choose 3 of the 9 remaining spaces that will contain R's, choose 2 of the remaining 6 spaces that will contain T's. Of the last 4 remaining positions choose one that will contain N, then choose one of the last 3 positions to contain D, then choose one of the last 2 positions to contain F and the H will fill the last position. The number of outcomes of this procedure is the number of rearrangements of the letters of the word GREATGRANDFATHER that do not have consecutive vowels. By the multiplication principle it is also equal to $\binom{12}{5}\binom{5}{2}\binom{11}{2}\binom{9}{3}\binom{6}{2}\cdot 4\cdot 3\cdot 2$.

(c) How many ways are there of rearranging the letters of the word GREATGRANDFA-THER such that the sequence of letters 'GRAND' appears consecutively somewhere in the word (in that order)?

Answer: The number of ways of rearranging the letters of the word GREATGRAND-FATHER such that the sequence of letters 'GRAND' appears consecutively is 29,937,600.

Solution: Imagine a sequence of 12 positions that will be filled with letters (and one of them will contain the word 'GRAND'). Choose 1 of the 12 positions to contain the word GRAND, next choose 1 of the remaining 11 positions for the last G, then 2 of the remaining 10 positions to contain the letter R, then 2 of the remaining 8 positions to contain the letter E, then 2 of the remaining 6 positions to contain the letter A, then 2 of the remaining 4 positions for the letter T, then 1 of the remaining 2 positions for the letter F, the final remaining position will contain the letter H. The number of outcomes of this procedure is the number of ways of rearranging the letter so that the word GRAND appear somewhere in the word. By the multiplication principle this is also equal to $12 \cdot 11 \cdot \binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2} \cdot 2$.

(4) Find all integers n such that $\phi(n) = 12$.
Answer: Only $n = 13, 21, 26, 28, 36$ or 42 have $\phi(n) = 12$.
Solution: By Theorem 6-2 in the book we have that if p^a divides n then $p^a - p^{a-1}$ divides $\phi(n)$. Since we are looking for all n such that $\phi(n) = 12 = 2 \cdot 2 \cdot 3$ we must have that if $2^a|n$ then $(2^a - 2^{a-1})|12$ and $0 \le a \le 3$. If $3^a|n$, then $(3^a - 3^{a-1})|12$ and $0 \le a \le 2$. If $7^a|n$ then $(7^a - 7^{a-1})|12$ and $0 \le a \le 1$. If $13^a|n$ then $(13^a - 13^{a-1})|12$ and $0 \le a \le 1$. All other primes have either $p^a - p^{a-1} > 12$ for all $a > 1$ or $p^a - p^{a-1}$ does not divide 12 (for example 5 and 11). Therefore we need only check the integers of the form $2^{a_2}3^{a_3}7^{a_7}13^{a_{13}}$ with $0 \le a_2 \le 3$, $0 \le a_3 \le 2$, $0 \le a_7 \le 1$, $0 \le a_{13} \le 1$ (i.e. 48 numbers at most) and this is a finite calculation which we find that only the ones listed above have $\phi(n) = 12$.

(5) Prove that if $r_1, r_2, \ldots, r_{\phi(m)}$ is a reduced residue system modulo m, and m is odd, then $r_1 + r_2 + \cdots + r_{\phi(m)} \equiv 0 \pmod{m}$. Hint: You can explain why this must be true by showing that $2r_1, 2r_2, \ldots, 2r_{\phi(m)}$ is also a reduced residue system and then explaining why the sum of any reduced residue system must be equivalent to the sum of any other reduced residue system modulo m.
Solution: Since m is odd then $gcd(2, m) = 1$. Therefore we know by p. 62 in the book that $2r_1, 2r_2, \ldots, 2r_{\phi(m)}$ is also a reduced residue system and that each of r_i is equivalent to $2r_{j_i}$ for some index j_i modulo m. Therefore $r_1 + r_2 + \cdots + r_{\phi(m)} \equiv 2r_{j_1} + 2r_{j_2} + \cdots + 2r_{j_{\phi(m)}} \equiv 2r_1 + 2r_2 + \cdots + 2r_{\phi(m)}$ $(mod\ m)$ and this implies that

$$r_1 + r_2 + \cdots + r_{\phi(m)} \equiv (2r_1 + 2r_2 + \cdots + 2r_{\phi(m)}) - (r_1 + r_2 + \cdots + r_{\phi(m)}) \equiv 0\ (mod\ m).$$

(6) Demonstrate the following identity by counting a set of objects in two different ways:

$$\binom{n}{4} + \binom{n-1}{3} + \binom{n-1}{2} = \binom{n+1}{4}$$

Solution: The right hand side of the equation represents the number of ways of choosing 4 numbers from the integers 1 through $n+1$. If the set does not contain $n+1$ then it contains 4 other numbers from the set 1 through n and there are $\binom{n}{4}$ of these sets. If the set does contain the number $n+1$ and not the number n then it also contains three other numbers from 1 to $n-1$ and there are $\binom{n-1}{3}$ such sets. If the set contains both $n+1$ and n then it also contains 2 numbers between 1 and $n-1$ and there are $\binom{n-1}{2}$ such sets. Every subset of size 4 of the numbers 1 through $n+1$ either does not contain the number $n+1$, or it contains $n+1$ but not n, or it contains n and $n+1$. By the addition principle the number of subsets with 4 numbers from 1 through $n+1$ is equal to $\binom{n}{4} + \binom{n-1}{3} + \binom{n-1}{2}$ and this is also $\binom{n+1}{4}$.

(7) Let $D_n = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_n}\right)$. Prove by induction that

$$D_n = \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ S = \{s_1, s_2, \ldots, s_{|S|}\}}} \frac{(-1)^{|S|}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}}$$

Example: $D_1 = \left(1 - \frac{1}{p_1}\right)$ and if $S \subseteq \{1\}$ then $S = \{\}$ or $S = \{1\}$. Therefore $D_1 = 1 + \frac{-1}{p_1}$. $D_2 = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)$ and if $S \subseteq \{1, 2\}$ then S is one of $\{\}, \{1\}, \{2\}, \{1, 2\}$. Therefore $D_2 = 1 + \frac{-1}{p_1} + \frac{-1}{p_2} + \frac{1}{p_1 p_2}$.

$D_3 = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)$ and if $S \subseteq \{1,2,3\}$ then S is one of $\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}$, $\{1,3\}, \{2,3\}, \{1,2,3\}$. Therefore $D_3 = 1 + \frac{-1}{p_1} + \frac{-1}{p_2} + \frac{-1}{p_3} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \frac{1}{p_2 p_3} + \frac{-1}{p_1 p_2 p_3}$.
Solution: The base of $n = 1, 2$ and 3 is give above. Notice that for $n \geq 2$ we have that $D_n = D_{n-1}\left(1 - \frac{1}{p_n}\right)$. Assume by induction that the identity holds for $n-1$. We have therefore that

$$\begin{aligned}
D_n &= \left(1 - \frac{1}{p_n}\right) \sum_{\substack{S \subseteq \{1,2,\ldots,n-1\} \\ S = \{s_1, s_2, \ldots, s_{|S|}\}}} \frac{(-1)^{|S|}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}} \\
&= \sum_{\substack{S \subseteq \{1,2,\ldots,n-1\} \\ S = \{s_1, s_2, \ldots, s_{|S|}\}}} \frac{(-1)^{|S|}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}} - \sum_{\substack{S \subseteq \{1,2,\ldots,n-1\} \\ S = \{s_1, s_2, \ldots, s_{|S|}\}}} \frac{(-1)^{|S|}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}} \frac{1}{p_n} \\
&= \sum_{\substack{S \subseteq \{1,2,\ldots,n-1\} \\ S = \{s_1, s_2, \ldots, s_{|S|}\}}} \frac{(-1)^{|S|}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}} + \sum_{\substack{S \subseteq \{1,2,\ldots,n-1,n\} \\ S = \{s_1, s_2, \ldots, s_{|S|}, n\}}} \frac{(-1)^{|S|+1}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}} \frac{1}{p_n} \\
&= \sum_{\substack{S \subseteq \{1,2,\ldots,n-1,n\} \\ S = \{s_1, s_2, \ldots, s_{|S|}\}}} \frac{(-1)^{|S|}}{p_{s_1} p_{s_2} \cdots p_{s_{|S|}}}
\end{aligned}$$

The last equality follows from the addition principle and because every subset of $\{1, 2, \ldots, n\}$ either contains the element n or it does not contain the element n and the first sum in the equation is the sum over the subsets which do not contain the element n and the second sum represents the sum over the subsets which do contain the element n.

(8) Let $p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k})$ with $p_{n,1} = 1$ and $p_{n,k} = 0$ if $k > n$. We give a table of $p_{n,k}$ for $n, k \leq 5$.

n/k	1	2	3	4	5	$\cdots$
1	1	0	0	0	0	
2	1	2	0	0	0	
3	1	6	6	0	0	
4	1	14	36	24	0	
5	1	30	150	240	120	
$\vdots$						

Show by induction that

$$p_{n,k} = k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots + (-1)^{k-1}\binom{k}{k-1}1^n \tag{1}$$

Solution: In the following proof we will use the identity

$$k\binom{k-1}{\ell} = k\frac{(k-1)!}{\ell!(k-\ell-1)!} = \frac{k!}{\ell!(k-\ell-1)!} = (k-\ell)\frac{k!}{\ell!(k-\ell)!} = (k-\ell)\binom{k}{\ell}.$$

For $n = 1$ we know that $p_{1,1} = 1$ and for $k > 1$ we have that

$$k - \binom{k}{1}(k-1) + \binom{k}{2}(k-2) - \binom{k}{3}(k-3) + \cdots + (-1)^{k-1}\binom{k}{k-1}1 =$$

$$k - \binom{k-1}{1}k + \binom{k-1}{2}k - \binom{k-1}{3}k + \cdots + (-1)^{k-1}\binom{k-1}{k-1}k =$$

$$k\left(\binom{k-1}{0} - \binom{k-1}{1} + \binom{k-1}{2} - \binom{k-1}{3} + \cdots + (-1)^{k-1}\binom{k-1}{k-1}\right) =$$

$$k(1-1)^{k-1} = 0 = p_{1,k}$$

We have therefore shown that equation (1) holds for $n = 1$. Now assume that equation (1) holds for all $n' < n$. Note that for $n > 1$ and for all k we have that $p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k})$ since by definition it holds for $k \leq n$ and for $k > n$, $p_{n,k} = 0 = k(0+0) = k(p_{n-1,k-1} + p_{n-1,k})$. We know therefore that

$$\begin{aligned}
p_{n,k} &= k(p_{n-1,k-1} + p_{n-1,k}) \\
&= k\Bigg((k-1)^{n-1} - \binom{k-1}{1}(k-2)^{n-1} + \binom{k-1}{2}(k-3)^{n-1} - \cdots + (-1)^{k-2}\binom{k-1}{k-2}1^{n-1} \\
&\quad + k^{n-1} - \binom{k}{1}(k-1)^{n-1} + \binom{k}{2}(k-2)^{n-1} - \binom{k}{3}(k-3)^{n-1} + \cdots + (-1)^{k-1}\binom{k}{k-1}1^{n-1}\Bigg) \\
&= k^n - k\cdot\left(\binom{k}{1} - \binom{k-1}{0}\right)(k-1)^{n-1} + k\cdot\left(\binom{k}{2} - \binom{k-1}{1}\right)(k-2)^{n-1} \\
&\quad - k\cdot\left(\binom{k}{3} - \binom{k-1}{2}\right)(k-3)^{n-1} + \cdots + (-1)^{k-1}k\cdot\left(\binom{k}{k-1} - \binom{k-1}{k-2}\right)1^{n-1} \\
&= k^n - k\cdot\binom{k-1}{1}(k-1)^{n-1} + k\cdot\binom{k-1}{2}(k-2)^{n-1} - k\cdot\binom{k-1}{3}(k-3)^{n-1} + \cdots \\
&\quad + (-1)^{k-1}k\cdot\binom{k-1}{k-1}1^{n-1} \\
&= k^n - \binom{k}{1}\cdot(k-1)\cdot(k-1)^{n-1} + \binom{k}{2}\cdot(k-2)\cdot(k-2)^{n-1} - \binom{k}{3}\cdot(k-3)\cdot(k-3)^{n-1} + \cdots \\
&\quad + (-1)^{k-1}\binom{k}{k-1}\cdot 1\cdot 1^{n-1} \\
&= k^n - \binom{k}{1}\cdot(k-1)^n + \binom{k}{2}\cdot(k-2)^n - \binom{k}{3}\cdot(k-3)^n + \cdots + (-1)^{k-1}\binom{k}{k-1}\cdot 1^n
\end{aligned}$$

Therefore equation (1) holds for all n.

(9) Let $p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k})$ with the initial conditions $p_{n,1} = 1$ and $p_{n,k} = 0$ for $k > n$. Show by induction on n that $p_{n,k}$ is equal to the number of ways of putting n different items in k different boxes such that there is at least one thing in each box.
Solution: Let $d_{n,k}$ be the number of ways of putting n different items in k different boxes such that there is at least one thing in each box. We know that for $n = 1$, $d_{1,1} = 1 = p_{1,1}$ and if $k > n$ then $d_{n,k} = 0 = p_{n,k}$ because if there are more boxes than items then there is no way that the items can be placed so that there is at least one item in each box.

Assume by induction that $p_{n',k'} = d_{n',k'}$ for all $1 \leq n' < n$. Now for $k \leq n$, either the n^{th} item is alone in one of the k boxes or the n^{th} item is not alone in one of the k boxes. If the n^{th} item is alone, then we choose which of the k boxes it is in and the remaining $k-1$ boxes are filled with the first $n-1$ items such that there is at least one item in each box (that is, by the multiplication principle in $kd_{n-1,k-1}$ ways). If the n^{th} item is not alone in a box then the first $n-1$ items are distributed in the k boxes such that there is at least one item in each box and the n^{th} item is in any one of the k boxes (that is, by the multiplication principle in $d_{n-1,k}k$ ways). Therefore by the addition principle we have that $d_{n,k} = kd_{n-1,k-1} + d_{n-1,k}k = k(d_{n-1,k-1} + d_{n-1,k})$. By the induction assumption we have that $d_{n,k} = k(d_{n-1,k-1} + d_{n-1,k}) = k(p_{n-1,k-1} + p_{n-1,k}) = p_{n,k}$. We conclude by induction that $d_{n,k} = p_{n,k}$ for all $n \geq 1$.