SOME CONNECTIONS BETWEEN ALGEBRAIC EXPRESSIONS AND SEQUENCES : PART II

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Given that

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + q^5 + \cdots$$

Prove the following algebraic equations have the given expansions:

(1)
$$\frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1}$$

(2)
$$\frac{1}{(1-q)^{k+1}} = \sum_{n\geq 0} \binom{n+k}{k} q^n$$

(3)
$$(1+aq)^n = \sum_{k\geq 0} \binom{n}{k} a^k q^k$$

(4)
$$\frac{1+q}{(1-q)^2} = 1 + 3q + 5q^2 + 7q^3 + 9q^4 + \cdots = \sum_{n\geq 0} (2n+1) q^n$$

(5)
$$\frac{2}{(1-q)^2} = 2 + 4q + 6q^2 + 8q^3 + 10q^4 + \cdots = \sum_{n\geq 0} (2n+2) q^n$$

(6)
$$\frac{q+q^2}{(1-q)^3} = q + 4q^2 + 9q^3 + 16q^4 + 25q^5 + \cdots = \sum_{n\geq 0} n^2 q^n$$

(7)
$$\frac{q+4q^2+q^3}{(1-q)^4} = q + 8q^2 + 27q^3 + 64q^4 + 125q^5 + \cdots = \sum_{n\geq 0} n^3 q^n$$

(8)
$$-ln(1-q) = q + q^2/2 + q^3/3 + q^4/4 + q^5/5 + \cdots = \sum_{n\geq 1} q^n/n$$

(9)
$$\frac{1}{1-q-q^2} = 1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 8\,q^5 + 13\,q^6 + \cdots = \sum_{n\geq 0} F_{n+1} q^n$$

(10)
$$\frac{1+2q}{1-q-q^2} = 1 + 3\,q + 4\,q^2 + 7\,q^3 + 11\,q^4 + 18\,q^5 + 29\,q^6 + 47\,q^7 + \cdots = \sum_{n\geq 0} L_{n+1} q^n$$

(11)
$$\frac{1-q}{1-3q+q^2} = 1 + 2q + 5q^2 + 13q^3 + 34q^4 + \cdots = \sum_{n\geq 0} F_{2n+1} q^n$$

(12)
$$\frac{1-q}{(1+q)(1-3q+q^2)} = 1 + q + 4q^2 + 9q^3 + 25q^4 + \cdots = \sum_{n\geq 0} F_{n+1}^2 q^n$$

(13)
$$\frac{1}{(1+q)(1-3q+q^2)} = 1 + 2q + 6q^2 + 15q^3 + 40q^4 + \cdots = \sum_{n\geq 0} F_{n+2}F_{n+1} q^n$$

(14)
$$\frac{2-q}{(1+q)(1-3q+q^2)} = 2 + 3q + 10q^2 + 24q^3 + \cdots = \sum_{n\geq 0} F_{n+3}F_{n+1} q^n$$

(15)
$$\frac{1}{1-3q-q^2} = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n\geq 0} F_{n+1}L_{n+1} q^n = \sum_{n\geq 0} F_{2n+2} q^n$$

(16)
$$\frac{1+4q-2q^2}{(1+q)(1-3q+q^2)} = 1 + 6q + 12q^2 + 35q^3 + 88q^4 + \cdots = \sum_{n\geq 0} F_{n+2}L_{n+1} q^n$$

(17)
$$\frac{1}{\sqrt{1-4q}} = 1 + 2\,q + 6\,q^2 + 20\,q^3 + 70\,q^4 + 252\,q^5 + 924\,q^6 + \cdots = \sum_{n\geq 0} \binom{2n}{n} q^n$$

(18)
$$\frac{1-\sqrt{1-4q}}{2q} = 1 + q + 2\,q^2 + 5\,q^3 + 14\,q^4 + 42\,q^5 + 132\,q^6 + \cdots = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} q^n$$

(19)
$$\frac{1}{\sqrt{1-4q}} \left(\frac{1-\sqrt{1-4q}}{2q}\right)^k = \sum_{n\geq 0} \binom{2n+k}{n} q^n$$

(20)
$$\left(\frac{1-\sqrt{1-4q}}{2q}\right)^k = \sum_{n\geq 0} \frac{k(2n+k-1)!}{n!(n+k)!} q^n$$