SOME CONNECTIONS BETWEEN ALGEBRAIC EXPRESSIONS AND COMBINATORIAL IDENTITIES

FEBRUARY 23, 2004

(1) Recall $\frac{1}{(1-q)^{k+1}} = \sum_{n\geq 0} \binom{n+k}{k} q^n$. Take the coefficient in the left and right hand side of

$$\frac{1}{1-q}\frac{1}{(1-q)^2} = \frac{1}{(1-q)^3}$$

and give a formula for $\sum_{i=1}^n i$.

(2) Recall $\frac{q+q^2}{(1-q)^3} = \sum_{n\geq 0} n^2 q^n$. Take the coefficient in the left and right hand side of

$$\frac{1}{1-q}\frac{q+q^2}{(1-q)^3} = \frac{q}{(1-q)^4} + \frac{q^2}{(1-q)^4}$$

and show #1 from p.6 of 'Number Theory.'

(3) Recall $\frac{q+4q^2+q^3}{(1-q)^4} = \sum_{n\geq 0} n^3 q^n$. Take the coefficient in the left and right hand side of

$$\frac{1}{1-q}\frac{q+4q^2+q^3}{(1-q)^4} = \frac{q}{(1-q)^5} + 4\frac{q^2}{(1-q)^5} + \frac{q^3}{(1-q)^5}$$

and show #2 from p.6 of 'Number Theory.'

(4) Recall that $\frac{d^2}{dq^2}\left(\frac{1}{1-q}\right) = \sum_{n\geq 0}(n+1)(n+2)q^n$. Take the coefficient in the left and right hand side of

$$\frac{1}{1-q}\frac{d^2}{dq^2}\left(\frac{1}{1-q}\right) = \frac{2}{(1-q)^3}$$

and show #4 from p.6 of 'Number Theory.'

(5) (a) What is the coefficient of q^n in $\frac{d}{dq}\left(-qln(1-q)\right)$?
(b) What is the coefficient of q^n in $\int_*^q \left(-ln(1-x)\right)dx$?

(6) Recall that $\frac{1}{1-q-q^2} = \sum_{n\geq 0} F_{n+1} q^n$ Take the coefficient of q^n in the left and right hand side of

$$\frac{1}{1-q}\frac{1}{1-q-q^2} = \frac{1}{q^2}\left(\frac{1}{1-q-q^2} - \frac{1}{1-q}\right)$$

and show #7 from p.6 of 'Number Theory.'

(7) (a) What is the coefficient of q^n in $\frac{1}{2}\left(\frac{1}{1-q-q^2} + \frac{1}{1+q-q^2}\right) = \frac{1-q^2}{(1-q-q^2)(1+q-q^2)}$?
(b) What is the coefficient of q^n in $\frac{1}{2q}\left(\frac{1}{1-q-q^2} - \frac{1}{1+q-q^2}\right) = \frac{1}{(1-q-q^2)(1+q-q^2)}$?
(c) Take the coefficient of q^{2n} in the left and right hand side of

$$\frac{1-q^2}{(1-q-q^2)(1+q-q^2)}\frac{1}{1-q^2} = \frac{1}{(1-q-q^2)(1+q-q^2)}$$

and show #8 from p.7 of 'Number Theory.'

(8) Recall $\frac{1+2q}{1-q-q^2} = \sum_{n\geq 0} L_{n+1} q^n$ Take the coefficient of q^n in the left and right hand side of the equation

$$\left(\frac{1+4q}{1-2q-4q^2}\right)\frac{q}{1-q} = \frac{1}{1-2q-4q^2} - \frac{1}{1-q}$$

and show #16 from p.7 of 'Number Theory.'

(9) Take the coefficient of q^m in the left and right hand side of the equation

$$\frac{1}{1-q}\frac{1}{(1-q)^{r+1}} = \frac{1}{(1-q)^{r+2}}$$

and show #5 from p.34 of 'Number Theory.'

(10) Take the coefficient of q^r in the left and right hand side of the equation

$$\frac{1}{1+q}(1+q)^n = (1+q)^{n-1}$$

(11) Take the coefficient of q^n in the left and right hand side of the equation

$$\frac{1}{1-(q+q^2)} = \sum_{k\geq 0}(q+q^2)^k = \sum_{k\geq 0} q^k(1+q)^k$$

and show #13 from p.35 of 'Number Theory.'

(12) Take the coefficient of q^{k+r} in the left and right hand side of the equation

$$(1+q)^n(1+q)^r = (1+q)^{n+r}$$

and show the identity that I gave you as a "Prove by counting in two different ways:" last term.