WORKSHEET VII: COMBINATORIAL IDENTITIES FROM GENERATING FUNCTIONS

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When you answer the following questions refer to the handout titled "Worksheet IV" with a list of useful generating functions.

(1) (a) Give a generating function which contains the expression
$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}$$
as one of the terms and which is related to the generating function for the sequence $\binom{2n}{k}$.

(b) Prove
$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}$$
Can you generalize this result?

(2) (a) Write down a generating function that has the coefficient of q^r equal to
$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r-k}$$

(b) Prove that
$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r}$$

(3) See "Number Theory" p.7 #8

(a) Write down a generating function such that the coefficient of q^{2n-1} is F_{2n} and the coefficient of q^{2n} is 0.

(b) Write down a generating function such that the coefficient of q^{2n} is F_{2n+1} and the coefficient of q^{2n+1} is 0.

(c) Is there an algebraic relation between these two expressions?

(d) Prove
$$F_1+F_3+F_5+\cdots+F_{2n-1}=F_{2n}$$

(4) See "Number Theory" p.7 #9

(a) Write down a generating function such that the coefficient of q^{2n} is F_{2n+1} and the coefficient of q^{2n+1} is 0.

(b) Subtract the generating function for $1+q^2+q^4+q^6+\cdots$.

(c) Is there a relationship between this generating function and the one for the generating function such that the coefficient of q^{2n-1} is F_{2n} and the coefficient of q^{2n} is 0?

(d) Prove
$$F_2+F_4+F_6+\cdots+F_{2n}=F_{2n+1}-1$$

(5) See "Number Theory" p.7 #10.
  (a) Give a generating function for F_{n+1}^2.
  (b) Give a generating function for $F_{n+1}F_{n+3}$.
  (c) Prove
$$F_{n+2}^2 - F_{n+1}F_{n+3} = (-1)^{n+1}$$
(6) See "Number Theory" p. 7 #11 and #12.
  (a) Give the generating function for $F_{n+1}F_{n+2}$.
  (b) Give the generating function for $F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{n+1}F_{n+2}$
  (c) Give the generating function for F_{2n}^2.
  (d) Find a relationship between the two functions and show:
$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$$
  and
$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n}F_{2n+1} = F_{2n+1}^2 - 1$$
  Hint: Try subtracting the generating function $1 + q^2 + q^4 + q^6 + \cdots = 1/(1-q^2)$.
(7) See "Number Theory" p. 6 #6.
  (a) Show that the coefficient of q^n in $q\frac{d}{dq}(-ln(1-q)/q)$ has coefficient of q^n equal to $\frac{n}{n+1}$.
  (b) Show that the coefficient of q^n in $ln(1-q)/q - ln(1-q) + 1 = ((1-q)ln(1-q)+q)/q$ is $\frac{1}{n(n+1)}$.
  (c) Show
$$\frac{1}{2\cdot 1} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$
  Hint (in case you forgot a little calculus) :
$$q\frac{d}{dq}(-ln(1-q)/q) = \frac{(1-q)\ln(1-q+q)}{(1-q)\,q}$$
(8) Take the coefficient of q^n in the equation
$$\frac{1}{(1-q)^a}\cdot\frac{1}{(1-q)^b} = \frac{1}{(1-q)^{a+b}}$$
(9) Take the coefficient of q^n in both sides of the equation
$$\frac{1}{\sqrt{1-4q}}\cdot\frac{1}{\sqrt{1-4q}} = \frac{1}{1-4q}$$
(10) Take the coefficient of q^n in both sides of the equation
$$\frac{1}{1+q}\cdot\frac{1}{1-3q+q^2} = \frac{1}{(1+q)(1-3q+q^2)}$$
(11) Take the coefficient of q^n in both sides of the equation
$$\frac{1}{(1-q)^a}\frac{1}{(1+q)^a} = \frac{1}{(1-q^2)^a}$$