

## WORKSHEET VIII: PARTITION GENERATING FUNCTIONS

MARCH 9, 2006

Match the description of the set of partitions with its generating function. Recall that a *partition* of  $n$  is a sum  $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$ . The order of the sum doesn't matter so to avoid confusion we assume that  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ . The  $\lambda_i$  are called the *parts* of the partition.  $r$  here is the number of parts of the partition or the *length* of the partition. The *sizes* of the parts are the values  $\lambda_i$ . The *size* of the partition is the sum of the sizes of all the parts (in this case  $n$ ). Parts are called *distinct* if they are not equal to each other. The *number of parts of a given size* refers to the number of times that a value appears as a part.

Note: There 14 generating functions and 15 descriptions listed below because two of the descriptions have the same generating function.

- (1) the number of partitions of  $n$
- (2) the number of partitions of  $n$  into exactly  $k$  parts
- (3) the number of partitions of  $n$  with parts of size  $k$  only
- (4) the number of partitions of  $n$  with parts of size less than or equal to  $k$
- (5) the number of partitions of  $n$  with distinct parts
- (6) the number of partitions of  $n$  with odd parts
- (7) the number of partitions of  $n$  with distinct odd parts
- (8) the number of partitions of  $n$  with even parts
- (9) the number of partitions of  $n$  with distinct even parts
- (10) the number of partitions of  $n$  into parts congruent to 1 or 4 modulo 5
- (11) the number of partitions of  $n$  with at most 4 parts of any given size
- (12) the number of partitions of  $n$  with (for each  $i$ ) the number of size  $i$  is less than  $i$ .
- (13) the number of partitions of  $n$  and for each  $i$ , if there is a part of size  $i$  then it occurs an odd number of times.
- (14) the number of partitions of  $n$  and for each  $i$ , the parts of size  $i$  occur an even number of times.
- (15) the number of partitions of  $n$  with only odd parts and the number of parts of any given size is even.

(a)

$$\prod_{i \geq 1} \frac{1}{1 - q^{2i-1}}$$

(b)

$$\frac{1}{1 - q^k}$$

(c)

$$\prod_{i=1}^k \frac{1}{1 - q^i}$$

1

(d)

$$\prod_{i \geq 1} \frac{1}{1 - q^{4i-2}}$$

(e)

$$\prod_{i \geq 1} (1 + q^i)$$

(f)

$$\prod_{i \geq 1} \left( 1 + \frac{q^i}{1 - q^{2i}} \right)$$

(g)

$$\prod_{i \geq 0} \frac{1}{(1 - q^{5i+1})(1 - q^{5i+4})}$$

(h)

$$\prod_{i \geq 1} \frac{1}{1 - q^i}$$

(i)

$$\prod_{i \geq 1} (1 + q^{2i-1})$$

(j)

$$q^k \prod_{i=1}^k \frac{1}{1 - q^i}$$

(k)

$$\prod_{i \geq 1} (1 + q^{2i})$$

(l)

$$\prod_{i \geq 1} \frac{1}{1 - q^{2i}}$$

(m)

$$\prod_{i \geq 1} \frac{1 - q^{5i}}{1 - q^i}$$

(n)

$$\prod_{i \geq 1} \frac{1 - q^{i^2}}{1 - q^i}$$