SOME FIBBONACCI GENERATING FUNCTIONS

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The following problems are all connected. Build up the library of generating functions you know by solving for the generating functions in the exercises below.

Recall $F(q) = \sum_{n\geq 0} F_{n+1}q^n = \frac{1}{1-q-q^2}$ and $L(q) = \sum_{n\geq 0} L_{n+1}q^n = \frac{1+2q}{1-q-q^2}$

(1) (a) Use the fact that $(A(q) + A(-q))/2 = \sum_{n\geq 0} a_{2n}q^{2n}$ to give a generating function for the odd Fibbonacci numbers $F_{odd}(q) = \sum_{n\geq 0} F_{2n+1}q^n$.
(b) Use the fact that $(A(q) - A(-q))/2 = \sum_{n\geq 0} a_{2n+1}q^{2n+1}$ to give a generating function for the even Fibbonacci numbers $F_{even}(q) = \sum_{n\geq 0} F_{2n+2}q^n$.

(2) Use the same tricks to find the generating functions for the even and odd Lucas numbers, $L_{odd}(q) = \sum_{n\geq 0} L_{2n+1}q^n$ and $L_{even}(q) = \sum_{n\geq 0} L_{2n+2}q^n$.

(3) (a) Use the following set of three formulas:

$$F_n^2 = F_n(F_{n-1} + F_{n-2}) = F_nF_{n-1} + F_nF_{n-2}$$
$$F_nF_{n+1} = F_n(F_n + F_{n-1}) = F_n^2 + F_nF_{n-1}$$
$$F_{n+2}F_n = F_n(F_{n+1} + F_n) = F_{n+1}F_n + F_n^2$$

to write down three equations with the generating functions $D^{(0)}(q) = \sum_{n\geq 0} F_{n+1}^2q^n$, $D^{(1)}(q) = \sum_{n\geq 0} F_{n+1}F_{n+2}q^n$, $D^{(2)}(q) = \sum_{n\geq 0} F_{n+1}F_{n+3}q^n$. Use those equations to solve for $D^{(0)}(q)$, $D^{(1)}(q)$, $D^{(2)}(q)$.
(b) Find a formula for $D^{(3)}(q) = \sum_{n\geq 0} F_{n+1}F_{n+4}q^n$ by replacing $F_{n+4} = F_{n+3} + F_{n+2}$ and expressing it in terms of $D^{(2)}(q)$ and $D^{(1)}(q)$.

(4) (a) The Lucas number satisfy the same recurrence as the Fibbonaci numbers from the previous problem. Use the same technique to find formulas for $E^{(0)}(q) = \sum_{n\geq 0} L_{n+1}^2q^n$, $E^{(1)}(q) = \sum_{n\geq 0} L_{n+1}L_{n+2}q^n$, $E^{(2)}(q) = \sum_{n\geq 0} L_{n+1}L_{n+3}q^n$.
(b) Find a formula for $E^{(3)}(q) = \sum_{n\geq 0} L_{n+1}L_{n+4}q^n$ by replacing $L_{n+4} = L_{n+3} + L_{n+2}$ and expressing it in terms of $E^{(2)}(q)$ and $E^{(1)}(q)$.

(5) (a) Use the results of the previous problems and the fact that $L_n = F_{n+1} + F_{n-1}$ for $n \geq 2$ to give a formula for the generating function $M^{(0)}(q) = \sum_{n\geq 0} F_{n+1}L_{n+1}q^n$.
(b) Use the generating functions $D^{(0)}(q)$, $D^{(1)}(q)$, $D^{(2)}(q)$ to give a formula for the generating function $M^{(1)}(q) = \sum_{n\geq 0} F_{n+2}L_{n+1}q^n$.
(c) Use the previous two problems and the fact that $L_{n+2} = L_{n+1} + L_n$ to find a formula for the generating function $M^{(-1)}(q) = \sum_{n\geq 0} F_{n+1}L_{n+2}q^n$

(6) (a) Find a formula for $F_{evensqr}(q) = \sum_{n\geq 0} F_{2n+2}^2q^n$.
(b) Find a formula for $F_{oddsqr}(q) = \sum_{n\geq 0} F_{2n+1}^2q^n$.

Record your answers below:

$$F_{odd}(q) = 1 + 2q + 5q^2 + 13q^3 + 34q^4 + \cdots = \sum_{n\geq 0} F_{2n+1}q^n =$$

$$F_{even}(q) = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n\geq 0} F_{2n+2}q^n =$$

$$L_{odd}(q) = 2 + 3q + 7q^2 + 18q^3 + 47q^4 + \cdots = \sum_{n\geq 0} L_{2n+1}q^n =$$

$$L_{even}(q) = 1 + 4q + 11q^2 + 29q^3 + 76q^4 + \cdots = \sum_{n\geq 0} L_{2n+2}q^n =$$

$$D^{(0)}(q) = 1 + q + 4q^2 + 9q^3 + 25q^4 + \cdots = \sum_{n\geq 0} F_{n+1}^2 q^n =$$

$$D^{(1)}(q) = 1 + 2q + 6q^2 + 15q^3 + 40q^4 + \cdots = \sum_{n\geq 0} F_{n+2}F_{n+1}q^n =$$

$$D^{(2)}(q) = 2 + 3q + 10q^2 + 24q^3 + 65q^4 + \cdots = \sum_{n\geq 0} F_{n+3}F_{n+1}q^n =$$

$$D^{(3)}(q) = 3 + 5q + 16q^2 + 39q^3 + 105q^4 + \cdots = \sum_{n\geq 0} F_{n+4}F_{n+1}q^n =$$

$$E^{(0)}(q) = 4 + q + 9q^2 + 16q^3 + 49q^4 + \cdots = \sum_{n\geq 0} L_{n+1}^2 q^n =$$

$$E^{(1)}(q) = 2 + 3q + 12q^2 + 28q^3 + 77q^4 + \cdots = \sum_{n\geq 0} L_{n+2}L_{n+1}q^n =$$

$$E^{(2)}(q) = 6 + 4q + 21q^2 + 44q^3 + 126q^4 + \cdots = \sum_{n\geq 0} L_{n+3}L_{n+1}q^n =$$

$$E^{(3)}(q) = 8 + 7q + 33q^2 + 72q^3 + 203q^4 + \cdots = \sum_{n\geq 0} L_{n+4}L_{n+1}q^n =$$

$$M^{(0)}(q) = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n\geq 0} F_{n+1}L_{n+1}q^n =$$

$$M^{(1)}(q) = 1 + 6q + 12q^2 + 35q^3 + 88q^4 + \cdots = \sum_{n\geq 0} F_{n+2}L_{n+1}q^n =$$

$$M^{(-1)}(q) = 3 + 4\,q + 14\,q^2 + 33\,q^3 + 90\,q^4 + \cdots = \sum_{n\geq 0} F_{n+1}L_{n+2}q^n =$$

$$F_{evensqr}(q) = 1 + 9q^2 + 64q^2 + 441q^3 + 3025q^4 + \cdots = \sum_{n\geq 0} F_{2n+2}^2 q^n =$$

$$F_{oddsqr}(q) = 1 + 4q^2 + 25q^2 + 169q^3 + 1156q^4 + \cdots = \sum_{n\geq 0} F_{2n+1}^2 q^n =$$

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55$$
$$L_1 = 2, L_2 = 1, L_3 = 3, L_4 = 4, L_5 = 7, L_6 = 11, L_7 = 18, L_8 = 29, L_9 = 47, L_{10} = 76$$

Using the equations that you found above, find a generating function proof of the following identities for $n \geq 0$:

(1)
$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n+1}F_{2n+2} = F_{2n+2}^2$$

(2)
$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_{n+1}F_{n+2}$$

(3)
$$F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$$

(4)
$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n+2}F_{2n+3} = F_{2n+3}^2 - 1$$

(5)
$$F_{n+1}L_{n+1} = F_{2n+2}$$

(6)
$$F_{n+2}^2 + 2F_{n+1}F_{n+2} = F_{2n+4}$$

(7)
$$F_{n+3}^2 - F_{n+1}^2 = F_{2n+4}$$

(8)
$$F_{n+2}^2 = F_{n+1}F_{n+3} + (-1)^{n+1}$$

(9)
$$F_{n+2}F_{n+3} = F_{n+1}F_{n+4} + (-1)^{n-1}$$

(10)
$$F_{n+2}L_{n+2} + F_{n+1}L_{n+1} = L_{2n+3}$$

(11)
$$F_{n+2}L_{n+2} - F_{n+1}L_{n+1} = F_{2n+3}$$

(12)
$$5(F_{n+1}^2 + F_{n+2}^2) = L_{n+1}^2 + L_{n+2}^2$$

(13)
$$5F_{n+1}^2 - L_{n+1}^2 = 4(-1)^n$$

(14)
$$L_{n+1}^2 - 2L_{2n+2} = -5F_n^2$$

(15)
$$F_{n+4} - F_{n+1} = 2F_{n+2}$$

(16)
$$F_{n+4} + F_{n+1} = 2F_{n+3}$$

(17)
$$F_{n+5} + F_{n+1} = 3F_{n+3}$$

(18)
$$F_{n+5} - F_{n+1} = L_{n+3}$$