PROOF OF A GENERATING FUNCTION EXPRESSION

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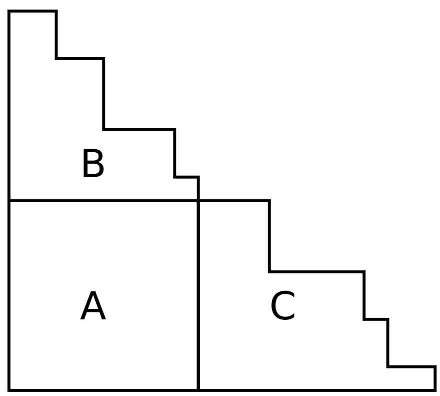
I wanted to give you an example of an generating function identity which we are able to justify in words now just using addition and multiplication principles of generating functions.

Look at the following diagram for a partition divided into pieces.



where we have

$A =$ a Durfee square = the largest square that fits inside of a partition

$B =$ a partition whose largest part is at most width of the Durfee square

$C =$ a partition with length at most the length of the Durfee square

Note that

$$q^{k^2}\left(\prod_{i=1}^{k}\frac{1}{(1-q^i)}\right)\left(\prod_{i=1}^{k}\frac{1}{(1-q^i)}\right)$$

is the generating function for triples consisting of: a Durfee square of size k, a partition of length less than or equal to k (which we showed in class to have generating function $\prod_{i=1}^{k}\frac{1}{1-q^i}$), and a partition with parts smaller than or equal to k (which we also showed in class to have a generating function $\prod_{i=1}^{k}\frac{1}{1-q^i}$).

Now the generating function for every partition has been shown in class to be $\prod_{i\geq 1}\frac{1}{1-q^i}$. We can also decompose the set of partitions into either empty the empty partition, or it has a Durfee square of size 1, or it has a Durfee square of size 2, or it has a Durfee square of size 3, etc. So by the addition principle of generating functions we have

$$\prod_{i\geq 1}\frac{1}{1-q^i} = 1 + \sum_{k\geq 1}\frac{q^{k^2}}{\prod_{i=1}^{k}(1-q^i)^2} \ .$$

This is an infinite identity but the partial expansions of of the generating function expression should be able to tell us what the first 10 or so terms are. So if we go to a computer and ask for an expansion of the first terms of both of these generating functions they should agree.

```
sage:  q = var('q')
sage:  taylor( mul(1/(1-q^(i+1)) for i in range(10)),q,0,11)
1 + q + 2*q^2 + 3*q^3 + 5*q^4 + 7*q^5 + 11*q^6 + 15*q^7 + 22*q^8
+ 30*q^9 + 42*q^10 + 55*q^11

sage:  taylor( 1+q/(1-q)^2+q^4/(1-q)^2/(1-q^2)^2+q^9/(1-q)^2/(1-q^2)^2/(1-q^3)^2,
q, 0, 11)
1 + q + 2*q^2 + 3*q^3 + 5*q^4 + 7*q^5 + 11*q^6 + 15*q^7 + 22*q^8
+ 30*q^9 + 42*q^10 + 56*q^11
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Notice that these two answers differ in the coefficient of q^{11} this is because the first expression $\prod_{i=1}^{10}\frac{1}{1-q^i}$ is not correct at the 11^{th} term because it doesn't have $1/(1-q^{11})$. The second expression should be correct up to the 16^{th} term.

Now, lets go one step further. We will also keep track of the length of the generating function as well as the size of the partition. That is we will write down the generating function for all partitions $\sum_{\lambda} z^{length(\lambda)}q^{size(\lambda)}$ in two different ways. The argument is essentially the same, it is just that we need to keep track of the power of z.

Consider the generating function for the partitions of consisting of only of parts of size k (rectangles of width k). It is equal to $\frac{1}{1-zq^k} = \sum_{r\geq 0} z^r q^{rk}$ because there is exactly one rectanglular partition of length r and width k and the size of that partition is rk. Every partition of n can be decomposed into a tuple consisting of the parts of size $1, 2, 3, \ldots, etc.$ By the multiplication principle the generating function for all partitions with a weight $z^{length}q^{size}$ is equal to

$$\prod_{k\geq 1}\frac{1}{1-zq^k} \ .$$

We can also decompose the set of partitions by the size of the Durfee square. Every partition is either empty, or it has a Durfee square of size 1 or it has a Durfee square of size 2, etc.

We can write down the generating function for the partitions with Durfee square of size exactly k. As above, we note that all partitions with Durfee square that is of size k consists of a square partition of length k (the A part on the diagram above), a partition whose largest part is of at most k (the B part on the diagram above), and a partition whose height is of most k (the C part on the diagram above).

The $k \times k$ Durfee square has generating function $z^k q^{k^2}$ because there is exactly one such square partition and its length is k and its size is k^2.

The generating function for the partitions whose largest part is at most k is $\prod_{i=1}^{k} \frac{1}{1-zq^i}$ because it consists of parts 1 through k and (as stated above) the generating function for just the parts of size i is $\frac{1}{1-zq^i}$.

Finally we note that the generating function for the partitions represented by the C part of the diagram is $\prod_{i=1}^{k} \frac{1}{1-q^i}$ because this piece of the partition does not change the total length of the partition and the generating function for partitions of length at most k counted by q^{size} is $\prod_{i=1}^{k} \frac{1}{1-q^i}$.

By the multiplication principle the number generating function for all of the partitions with Durfee square of size $k \times k$ is

$$z^k q^{k^2} \prod_{i=1}^{k} \frac{1}{1-zq^i} \prod_{i=1}^{k} \frac{1}{1-q^i}$$

Therefore by the addition principle the generating function for the number of partitions with weight $z^{length} q^{size}$ is also equal to

$$1 + \sum_{k \geq 1} z^k q^{k^2} \prod_{i=1}^{k} \frac{1}{1-zq^i} \prod_{i=1}^{k} \frac{1}{1-q^i} \ .$$

We have therefore shown

$$\prod_{k \geq 1} \frac{1}{1-zq^k} = 1 + \sum_{k \geq 1} z^k q^{k^2} \prod_{i=1}^{k} \frac{1}{1-zq^i} \prod_{i=1}^{k} \frac{1}{1-q^i} \ .$$

So now lets show an example of our refined result. We note below that in the expansion of the series below that the two expressions differ in the coefficient of q^{11} because our product $\prod_{i=1}^{10} \frac{1}{1-zq^i}$ doesn't quite go high enough to get the q^{11} term right.

In the calculation below we have parameters q and z and if we look at for instance the coefficient of $z^5 q^8$ we see the value of 3. This is because $(4,1,1,1,1), (3,2,1,1,1), (2,2,2,1,1)$ are the only three partitions of 8 of length 5.

```
sage:  (z,q) = var('z q')
sage:  taylor( mul(1/(1-z*q^(i+1)) for i in range(10)),q,0,11)
(z^11 + z^10 + 2*z^9 + 3*z^8 + 5*z^7 + 7*z^6 + 10*z^5 + 11*z^4 + 10*z^3 + 5*z^2)*q^11
+ (z^10 + z^9 + 2*z^8 + 3*z^7 + 5*z^6 + 7*z^5 + 9*z^4 + 8*z^3 + 5*z^2 + z)*q^10 + (z^9
+ z^8 + 2*z^7 + 3*z^6 + 5*z^5 + 6*z^4 + 7*z^3 + 4*z^2 + z)*q^9 + (z^8 + z^7 + 2*z^6
+ 3*z^5 + 5*z^4 + 5*z^3 + 4*z^2 + z)*q^8 + (z^7 + z^6 + 2*z^5 + 3*z^4 + 4*z^3 + 3*z^2
+ z)*q^7 + (z^6 + z^5 + 2*z^4 + 3*z^3 + 3*z^2 + z)*q^6 + (z^5 + z^4 + 2*z^3 + 2*z^2
+ z)*q^5 + (z^4 + z^3 + 2*z^2 + z)*q^4 + (z^3 + z^2 + z)*q^3 + (z^2 + z)*q^2 + q*z
+ 1

sage:  taylor( 1+z*q/(1-q)/(1-z*q)+z^2*q^4/(1-q)/(1-q^2)/(1-z*q)/(1-z*q^2)
+z^3*q^9/(1-q)/(1-q^2)/(1-q^3)/(1-z*q)/(1-z*q^2)/(1-z*q^3), q, 0, 11)
(z^11 + z^10 + 2*z^9 + 3*z^8 + 5*z^7 + 7*z^6 + 10*z^5 + 11*z^4 + 10*z^3 + 5*z^2 + z)*q^11
+ (z^10 + z^9 + 2*z^8 + 3*z^7 + 5*z^6 + 7*z^5 + 9*z^4 + 8*z^3 + 5*z^2 + z)*q^10 + (z^9
+ z^8 + 2*z^7 + 3*z^6 + 5*z^5 + 6*z^4 + 7*z^3 + 4*z^2 + z)*q^9 + (z^8 + z^7 + 2*z^6
+ 3*z^5 + 5*z^4 + 5*z^3 + 4*z^2 + z)*q^8 + (z^7 + z^6 + 2*z^5 + 3*z^4 + 4*z^3 + 3*z^2
+ z)*q^7 + (z^6 + z^5 + 2*z^4 + 3*z^3 + 3*z^2 + z)*q^6 + (z^5 + z^4 + 2*z^3 + 2*z^2
+ z)*q^5 + (z^4 + z^3 + 2*z^2 + z)*q^4 + (z^3 + z^2 + z)*q^3 + (z^2 + z)*q^2 + q*z
+ 1
```