WIDGETS AND DOODLES II

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Say that $a_0, a_1, a_2, \ldots$ is a sequence of non-negative integers where a_n represents the number of "widgets of type n." Assume similarly that $b_0, b_1, b_2, \ldots$ is a sequence where b_n represents the number of "doodles of type n." Each of the following descriptions for a set of elements can be counted using the addition and multiplication principle. For each of the sets below, decompose the description and give an expression in terms the variables a_i and b_i and other arithmetic operations.

(1) the set of things which are either widgets or doodles of type at most n
(2) the set of pairs consisting of a widget of type at most n and a doodle whose type is twice the type of the widget
(3) the set of pairs consisting of an integer 1 through n and an object of the same type as the integer which is either a widget if the integer is even or a doodle if the integer is odd
(4) the set of pairs consisting of a widgets of type n and a doodle which is a type strictly smaller than the type of the widget
(5) the set of pairs consisting of a widgets of type at most n and a doodle which is a type strictly smaller than the type of the widget
(6) the set of pairs consisting of a widget of type at most n and a doodle whose type is at most half the type of the widget
(7) the set of pairs consisting of an integer between 1 and n and a widget of the same type as the integer
(8) the set of pairs consisting of an integer between 1 and n and a widget of type smaller than the integer
(9) the set of pairs consisting of a widget of type less than or equal to n and an integer between 1 and the type of the widget
(10) the set of pairs consisting of a widget of type n and a doodle of type less than or equal to n
(11) the set of pairs consisting of an integer 1 through n and a widget which is of type in between the integer and n
(12) the set of pairs whose first element is a subset of the integers 1 through n of size 2 and a widget of type less than or equal to n but greater than or equal to the larger of the two numbers in the set
(13) the set of set of triples whose first two elements are widgets whose sum of the types is n and a doodle that is type less than or equal to n
(14) the set of triple consisting of two widgets of type n that are different from each other and a doodle of type n
(15) the set of collections of 3 different widgets of type n (order does not matter)

Write a combinatorial description of a set by combining the sets 'widgets' and 'doodles' of various types such that the number of elements in the set is equal to the following expressions. Your description can be similar to the ones above where we describe "the set of ..." or it can begin with a phrase like "the number of ways of ..."

(1) $a_1^2 + a_2^2 + \cdots + a_n^2$
(2) $a_0b_0 + a_1b_1 + a_2b_2 + \cdots + a_nb_n$
(3) $1 + a_1 + a_2^2 + \cdots + a_n^n$
(4) $b_n^n + b_{n-1}^{n-1}a_1 + b_{n-2}^{n-2}a_2^2 + \cdots b_1a_{n-1}^{n-1} + a_n^n$
(5) $b_n^2a_0 + b_{n-1}^2a_1 + b_{n-2}^2a_2 + \cdots b_0^2a_n$
(6) $(a_0 + a_1)b_0 + (a_1 + a_2)b_1 + (a_2 + a_3)b_2 + \cdots + (a_n + a_{n+1})b_n$
(7) $\binom{2}{2}a_2 + \binom{3}{2}a_3 + \binom{4}{2}a_4 + \cdots + \binom{n}{2}a_n$
(8) $\binom{n}{0}a_0 + \binom{n}{1}a_1 + \binom{n}{2}a_2 + \cdots + \binom{n}{n}a_n$
(9) $\binom{n}{0}a_1 + \binom{n}{1}a_3 + \binom{n}{2}a_5 + \cdots + \binom{n}{n}a_{2n+1}$
(10) $(a_1 + a_3 + a_5 + \cdots + a_{2n+1})(b_0 + b_2 + b_4 + \cdots + b_{2n})$
(11) $a_n!$
(12) $a_1^2b_n + a_2^2b_{n-1} + \cdots + a_n^2b_1$
(13) $b_0^2a_0^2 + b_1^2a_2^2 + b_2^2a_4^2 + \cdots + b_n^2a_{2n}^2$
(14) $b_0a_1 + b_1a_3 + b_2a_5 + \cdots + b_na_{2n+1}$
(15) $a_0b_0 + (a_0 + a_1)b_1 + (a_0 + a_1 + a_2)b_2 + \cdots + (a_0 + a_1 + \cdots + a_n)b_n$
(16) $a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_1 + b_2 + \cdots + b_n) + a_2(b_2 + b_3 + \cdots + b_n) + \cdots + a_nb_n$
(17) $a_n^{b_n}$
(18) $a_1^{b_1} + a_2^{b_2} + \cdots + a_n^{b_n}$