Examples of generating functions

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For each of the following generating functions give a short proof (or sketch) that the formula matches the sequence.

(1) A sequence of 1s

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + q^5 + \cdots$$

(2) a sequence of k 1's followed by 0's

$$\frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1}$$

(3) binomial coefficients, top number changing

$$\frac{1}{(1-q)^{k+1}} = \binom{k}{k} + \binom{k+1}{k} q + \binom{k+2}{k} q^2 + \binom{k+3}{k} q^3 + \cdots = \sum_{n \geq 0} \binom{n+k}{k} q^n$$

(4) binomial coefficients, bottom number changing

$$(1+q)^n = 1 + \binom{n}{1} q + \binom{n}{2} q^2 + \binom{n}{3} q^3 + \cdots = 1 + \binom{n}{1} q + \binom{n}{2} q^2 + \cdots + \binom{n}{n} q^n = \sum_{k \geq 0} \binom{n}{k} q^k$$

(5) odd integers

$$\frac{1+q}{(1-q)^2} = 1 + 3q + 5q^2 + 7q^3 + 9q^4 + \cdots = \sum_{n \geq 0} (2n+1) q^n$$

(6) even integers

$$\frac{2}{(1-q)^2} = 2 + 4q + 6q^2 + 8q^3 + 10q^4 + \cdots = \sum_{n \geq 0} (2n+2) q^n$$

(7) squares of integers

$$\frac{q+q^2}{(1-q)^3} = q + 4q^2 + 9q^3 + 16q^4 + 25q^5 + \cdots = \sum_{n\geq 0} n^2 q^n$$

(8) cubes of integers

$$\frac{q+4q^2+q^3}{(1-q)^4} = q + 8q^2 + 27q^3 + 64q^4 + 125q^5 + \cdots = \sum_{n\geq 0} n^3 q^n$$

(9) one over an integer

$$-ln(1-q) = q + q^2/2 + q^3/3 + q^4/4 + q^5/5 + \cdots = \sum_{n\geq 1} q^n/n$$

(10) Fibbonacci numbers

$$\frac{1}{1-q-q^2} = 1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 8\,q^5 + 13\,q^6 + \cdots = \sum_{n\geq 0} F_{n+1} q^n$$

(11) Lucas numbers

$$\frac{1+2q}{1-q-q^2} = 1 + 3\,q + 4\,q^2 + 7\,q^3 + 11\,q^4 + 18\,q^5 + 29\,q^6 + 47\,q^7 + \cdots = \sum_{n\geq 0} L_{n+1} q^n$$

(12) odd Fibbonacci numbers

$$\frac{1-q}{1-3q+q^2} = 1 + 2q + 5q^2 + 13q^3 + 34q^4 + \cdots = \sum_{n\geq 0} F_{2n+1} q^n$$

(13) even Fibbonacci numbers

$$\frac{1}{1-3q+q^2} = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n\geq 0} F_{2n+2} q^n$$