Some number theory and combinatorics questions

January 12, 2012

(1) How many 2×2 invertible matrices are there mod p?
(2) How many different full house hands can be made from a 52 card deck if the A of spades is thrown out?
(3) Find r and s such that $gcd(119, 315) = 119r + 315s$,
(4) Alice and Bob wish to set up a public key cryptosystem. Their first step is to agree on a public modulus $p = 17$ and the primitive root $a = 11$. Alice publishes her public key as the number 5 (remember it is the primitive root raised to her secret key) and Bob publishes 14 as his public key. What is the common key between Alice and Bob ($a^{\text{secret key for Alice} \cdot \text{secret key for Bob}}$). The powers of 3 *mod* 17 are

$$3^1 = 3, 3^2 = 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$$

(5) Find an integer x such that

$$202x \equiv 33 \ (mod\ 431)$$

(6) Determine by computing a Jacobi/Legendre symbol if

$$x^2 + 14x \equiv 194 \ (mod\ 389)$$

has a solution.

(7) How many ways are there of rolling 6 indistinguishable six-sided dice so that three of the dice are showing the same value and three are showing different values? answer: 60
(8) How many ways are there of rolling 6 indistinguishable six-sided dice so that exactly 3 different values are showing? answer: 200
(9) How many ways are there of rolling 12 indistinguishable six-sided dice so that exactly half are odd and half are even? answer: 784
(10) How many ways are there of rolling 6 different colored (and hence distinguished) six-sided dice so that the sum of the dice is 18? answer: 3,431
(11) How many ways are there of rolling 6 different colored (and hence distinguished) six-sided dice so that three of the dice are showing the same value and three are showing different values? answer: 7,200
(12) How many ways are there of rolling 6 different colored (and hence distinguished) six-sided dice so that exactly 3 different values are showing? answer: 10,800
(13) How many ways are there of rolling 12 different colored (and hence distinguished) six-sided dice so that exactly half are odd and half are even? answer: 491,051,484
(14) How many ways are there of placing 14 indistinguishable balls in 6 boxes so that the first 2 boxes have at most six of the balls. answer: 8,526
(15) How many ways are there of placing 14 indistinguishable balls in 6 boxes so that the first 2 boxes have at least half of the balls. answer: 3,102
(16) How many ways are there of making change for 35 cents using quarters, nickels, pennies (all Canadian) and U.S nickels and U.S. pennies? answer: 482
(17) How many ways are there of picking 22 marbles from 18 red ones, 5 blues and 10 pinks? answer: 56
(18) How many ways can 5 red balls, 5 blue balls, 10 green balls and 3 yellow balls be ordered so that green balls are are not adjacent? answer: 72,144,072
(19) How many ways can 5 red balls, 5 blue balls, 4 green balls and 3 yellow balls be ordered so that no more than 3 red balls are adjacent? answer: 166,846,680
(20) How many ways can 5 red balls, 5 blue balls, 4 green balls and 3 yellow balls be ordered so that exactly 2 blue balls are adjacent? answer: 47,567,520