A FEW WORDS ABOUT TELESCOPING SUMS

MIKE ZABROCKI

Say that you want to prove an identity of the form

\[ p(1) + p(2) + p(3) + \cdots + p(n) = q(n) \]

where \( p(x) \) and \( q(x) \) are expressions and \( n \) is a non-negative integer. We must have that \( q(0) = 0 \) since the left hand side of the equation will be the empty sum when \( n = 0 \). One way to go about this is to show that

\[ q(r) - q(r - 1) = p(r) \]

for all \( r \geq 0 \), and then write down

\[ q(1) - q(0) = p(1) \]
\[ q(2) - q(1) = p(2) \]
\[ q(3) - q(2) = p(3) \]
\[ \vdots \]
\[ q(n - 1) - q(n - 2) = p(n - 1) \]
\[ q(n) - q(n - 1) = p(n) \]

Now the sum of the expressions on the right-hand side of this equation is

\[ p(1) + p(2) + p(3) + \cdots + p(n - 1) + p(n) \]

and the sum of the expressions on the left-hand side of this equation is

\[ (q(1) - q(0)) + (q(2) - q(1)) + (q(3) - q(2)) + \cdots + (q(n - 1) - q(n - 2)) + (q(n) - q(n - 1)) = q(n) - q(0) = q(n) \]

We conclude therefore that

\[ p(1) + p(2) + p(3) + \cdots + p(n - 1) + p(n) = q(n) \]

Example There are lots of ways of proving the following identity.

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}. \]

Since \( \frac{r(r+1)}{2} - \frac{(r-1)r}{2} = r \), we have

\[ \frac{1 \cdot 2}{2} - \frac{0 \cdot 1}{2} = 1 \]
\[
\begin{align*}
2 \cdot 3 \cdot 2 - 1 \cdot 2 \cdot 2 &= 2 \\
3 \cdot 4 \cdot 2 - 2 \cdot 3 \cdot 2 &= 3
\end{align*}
\]
\[
\vdots
\]
\[
\left(\frac{n-1}{2}\right) \cdot n - \frac{(n-2)(n-1)}{2} = n - 1
\]
\[
\frac{n \cdot (n+1)}{2} - \frac{(n-1) \cdot n}{2} = n
\]
The sum of the terms on the left hand side of these equations is \(\frac{n(n+1)}{2}\) and the sum of the terms on the right hand side of these equation is \(1 + 2 + 3 + \cdots + (n-1) + n\), therefore they are equal.

**Example** Define the Fibonacci sequence by \(F_0 = 1\), \(F_1 = 1\), and for \(n \geq 0\), \(F_{n+2} = F_{n+1} + F_n\). Say that we want to show that
\[
F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - F_1,
\]
or in words “The sum of the first \(n\) Fibonacci numbers indexed by even \(n\) is the next Fibonacci number indexed by odd \(n\)” So we know that for \(r \geq 1\), \(F_{2r+1} - F_{2r-1} = F_{2r} + F_{2r-1} - F_{2r-1} = F_{2r}\). Therefore
\[
F_3 - F_1 = F_2
\]
\[
F_5 - F_3 = F_4
\]
\[
F_7 - F_5 = F_6
\]
\[
\vdots
\]
\[
F_{2n-1} - F_{2n-3} = F_{2n-2}
\]
\[
F_{2n+1} - F_{2n-1} = F_{2n}
\]
Since the sum of the left hand side of these equations is \(F_{2n+1} - F_1 = F_{2n+1} - F_0\) and the sum of the right hand side of this equation is \(F_2 + F_4 + F_6 + \cdots + F_{2n}\), we conclude that
\[
F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}.
\]

**Example** Here is a general identity that can be fairly useful:
\[
1 \cdot 2 \cdots k + 2 \cdot 3 \cdots (k+1) + 3 \cdot 4 \cdots (k+2) + \cdots + n \cdot (n+1) \cdots (n+k-1)
\]
\[
= n \cdot (n+1) \cdots (n+k)/(k+1)
\]
Observations: (1) if \(k = 1\), then this identity reduces to \(1 + 2 + 3 + \cdots + n = n(n+1)/2\). (2) if \(k = 2\), then this identity reduces to \(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + n \cdot (n+1) = n(n+1)(n+2)/3\).
There is shorthand notation that makes this sum easier to work with. Let \( (a)_k = a(a + 1)(a + 2) \cdots (a + k - 1) \), then the identity becomes
\[
(1)_k + (2)_k + (3)_k + \cdots + (n)_k = (n)_{k+1}/(k + 1)
\]

We note that
\[
r \cdot (r + 1) \cdots (r + k)/(k + 1) = r \cdot (r + 1) \cdots (r + k - 1)((r + k) - (r - 1))/(k + 1) = r \cdot (r + 1) \cdots (r + k - 1).
\]

Therefore we have
\[
1 \cdot 2 \cdot (k + 1)/((k + 1) - 0 \cdot 1 \cdots k/(k + 1) = 1 \cdot 2 \cdots k
\]
\[
2 \cdot 3 \cdot (k + 2)/(k + 1) - 1 \cdot 2 \cdot (k + 1)/(k + 1) = 2 \cdot 3 \cdots (k + 1)
\]
\[
3 \cdot 4 \cdot (k + 3)/(k + 1) - 2 \cdot 3 \cdot (k + 2)/(k + 1) = 3 \cdot 4 \cdots (k + 2)
\]
\[
\vdots
\]
\[
(n-1) \cdot n \cdots (n+k-1)/(k+1) - (n-2) \cdot (n-1) \cdots (n+k-2)/(k+1) = (n-1) \cdot n \cdots (n+k-2)
\]
\[
n \cdot (n+1) \cdots (n+k)/(k+1) - (n-1) \cdot (n-2) \cdots (n+k-1)/(k+1) = n \cdot (n+1) \cdots (n+k-1)
\]

The sum of the entries on the left hand side of these equalities is \( n \cdot (n+1) \cdots (n+k)/(k+1) \) and the sum of the entries on the right hand side of these equalities is
\[
1 \cdot 2 \cdots k + 2 \cdot 3 \cdots (k + 1) + 3 \cdot 4 \cdots (k + 2) + \cdots + n \cdot (n+1) \cdots (n+k-1),
\]

therefore the two expressions are equal.

One final observation: It is always possible to express \( n^k \) as a sum in the notation \( (n)_r \).
\[
n^1 = (n)_1, \quad n^2 = (n)_2 - (n)_1, \quad n^3 = (n)_3 - 3(n)_2 + (n)_1, \quad n^4 = (n)_4 - 6(n)_3 + 7(n)_2 - (n)_1.
\]

This can be used to give a sum of \( 1^k + 2^k + 3^k + \cdots + n^k \). The coefficients in this expansion are known as the Stirling numbers of the second kind.