

F. FRANKLIN'S PROOF OF EULER'S PENTAGONAL NUMBER THEOREM

ABSTRACT. The 18th century mathematician Leonard Euler discovered a simple formula for the expansion of the infinite product $\prod_{i \geq 1} 1 - q^i$. In 1881, one of the first American mathematicians found an elegant combinatorial proof of this identity.

Proposition 1. (*Euler's pentagonal number theorem*)

$$(1) \quad \prod_{i \geq 1} 1 - q^i = 1 + \sum_{m \geq 1} (-1)^m \left(q^{\frac{m(3m-1)}{2}} + q^{\frac{m(3m+1)}{2}} \right)$$

There is a clever proof of this proposition that comes from one of the first American mathematicians F. Franklin [4]. The proof uses a technique which is fairly ubiquitous in algebraic combinatorics, to show that terms in a sum cancel associate a combinatorial object to each term in the sum and then show that they cancel by producing a map which sends an element with positive weight to a term with negative weight.

There are several other accounts of this proof: [2], [6], [7], [9].


We will need to talk about partitions as a combinatorial object. λ is a partition if it is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ with $\lambda_1 \geq \lambda_2 \geq \dots \lambda_{\ell(\lambda)}$ where we use the notation $\ell(\lambda)$ to represent the number of parts of λ . The symbol $|\lambda|$ will represent the size of the partition so that $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_{\ell(\lambda)}$. λ is a strict partition if in addition $\lambda_1 > \lambda_2 > \dots \lambda_{\ell(\lambda)}$.

There is a way of graphically representing a partition with rows of boxes. A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ is represented by a row of λ_1 boxes below a row of λ_2 boxes below a row of λ_3 boxes etc. Each of these rows of cells will be left justified. For example the partition $(4, 4, 3, 1, 1)$ is represented by the following diagram:



Example 1. We note that the left hand side of this equation is the generating function for all strict partitions (partitions where all parts are distinct) weighted with $(-1)^{\ell(\lambda)} q^{|\lambda|}$. That is,

$$(2) \quad \prod_{i \geq 1} 1 - q^i = \sum_{\substack{\lambda \text{ strict} \\ 1}} (-1)^{\ell(\lambda)} q^{|\lambda|}$$

This follows by observing that to determine the coefficient of q^n by expansion of the product on the left we have a contribution of $(-1)^k q^{\lambda_1 + \lambda_2 + \dots + \lambda_k}$ for every sequence $(\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_i > \lambda_{i+1}$ for $1 \leq i < k$. Below we expand the terms of this generating function through degree 8. For example, a term of the form $(-q^4)(-q^2)$ is represented by the picture  and we record the weight of $+q^6$ just below the picture.

$$\begin{array}{cccccccccccc}
 \cdot & \square & \square\square & \begin{array}{c} \square \\ \square\square \end{array} & \square\square\square & \begin{array}{c} \square \\ \square\square \\ \square\square \end{array} & \square\square\square\square & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} \\
 1 & -q & -q^2 & +q^3 & -q^3 & +q^4 & -q^4 & +q^5 & +q^5 & -q^5 & -q^6 & +q^6 \\
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 & +q^6 & -q^6 & -q^7 & +q^7 & +q^7 & +q^7 & -q^7 & -q^8 \\
 \\
 \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \dots & \\
 & -q^8 & +q^8 & +q^8 & +q^8 & -q^8 & +\dots
 \end{array}$$

Now we notice that all of the terms cancel except for the ones stated in the theorem, that is we have

$$\prod_{i \geq 1} (1 - q^i) = 1 - q - q^2 + q^5 + q^7 + \dots$$

In fact, we will show that one way of looking at this expression is to observe terms which survive are those that correspond to the following pictures:

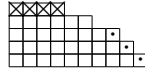
$$\begin{array}{cccccccc}
 \prod_{i \geq 1} (1 - q^i) = & \cdot & \square & \square\square & \begin{array}{c} \square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} \\
 & 1 & -q & -q^2 & +q^5 & +q^7 & -q^{12} & -q^{15} \\
 \\
 \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \begin{array}{c} \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \\ \square\square \end{array} & \dots & \\
 & +q^{22} & +q^{26} & -q^{35} & -q^{40} & \dots
 \end{array}$$

From the image in this example one might think that the theorem would be better named the *trapezoidal* number theorem. There is a reason that the numbers $m(3m - 1)/2$ are referred to as pentagonal numbers and if $m \rightarrow -m$ then the pentagonal number is transformed to $\rightarrow -m(-3m - 1)/2 = m(3m + 1)/2$. Observe the picture below how a sequence of pentagons have exactly $m(3m - 1)/2$ points in them (and this continues for $m > 3$).

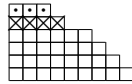
Proof. To show that this proposition holds we show that there is an involution ϕ on the strict partitions λ of n such that $\phi(\lambda)$ is also a partition of n and the length of $\phi(\lambda)$ will have length either one smaller or one larger than that of λ . This means that if the weight of a strict partition is $(-1)^{\ell(\lambda)}q^{|\lambda|}$ then the weight of $\phi(\lambda)$ is $-(-1)^{\ell(\lambda)}q^{|\lambda|}$ and so this term corresponding to $\phi(\lambda)$ will cancel with the term corresponding to λ . This involution will fail to ‘work’ for the partitions of the form $(2m - 1, 2m - 2, \dots, m)$ which are of size $2m^2 - \frac{(m+1)m}{2} = \frac{m(3m-1)}{2}$ and $(2m, 2m - 1, \dots, m + 1)$ which are of size $2m^2 - \frac{(m-1)m}{2} = \frac{m(3m+1)}{2}$.

For a strict partition λ we will let r equal to the smallest part of λ ($r = \lambda_{\ell(\lambda)}$) and let s equal the number of parts which are consecutive at the beginning of the partition. In other words s is the largest integer such that $(\lambda_1, \lambda_2, \dots, \lambda_s) = (\lambda_1, \lambda_1 - 1, \dots, \lambda_1 - s + 1)$.

If $s \neq \ell(\lambda)$ and $r > s$ then we will let $\phi(\lambda)$ equal the partition $(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_s - 1, \lambda_{s+1}, \dots, \lambda_{\ell(\lambda)}, s)$. That is, if the diagram for the partition looks something like the following where there is an \times in each of the cells corresponding to r and a dot in the cells corresponding to s

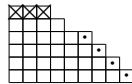


then $\phi(\lambda)$ will be the partition with the diagonal of s cells filled with a dot moved to the top row of the partition.

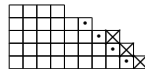


$\phi(\lambda)$ is still a strict partition and it has the property that the longest string of consecutive parts at the beginning of the partition is greater than or equal to s .

If $s \neq \ell(\lambda)$ and $r \leq s$ then we will let $\phi(\lambda)$ equal to the partition $(\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_r + 1, \lambda_{r+1}, \dots, \lambda_{\ell(\lambda)})$. For example, if our diagram is similar to the one below with the cells marked with an \times representing the row of size r and those marked with the \cdot represent the cells which correspond to the s consecutive parts at the beginning of the partition.



The partition corresponding to $\phi(\lambda)$ is then represented by the following picture.



Notice that it is also possible that $s = \ell(\lambda)$ and we consider this case separately because we need that r is at least 2 more cells than s does before we can move the s cells to the top row. In this case if $r > s + 1$ then we will remove the s cells along the diagonal and turn them

into the shortest row so that $\phi(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_s - 1, s)$. For example we have the picture on the left will be transformed to the one on the right.



If $s = \ell(\lambda)$ and $r < s$ then it is still possible to move the shortest row of λ to the first r rows. We will set $\phi(\lambda) = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_r + 1, \lambda_r, \dots, \lambda_{\ell(\lambda)-1})$, this corresponds to the case when we have a partition of the form of the one below.



If we describe what is happening to the diagram the map ϕ does one of two things, either it removes the smallest row of $r = \lambda_{\ell(\lambda)}$ cells of the partition and places one cell more in each of the first r rows (in the case that $r < s$ or $r = s$ and $s < \ell(\lambda)$) or it removes one cell from each of the first s rows and adds a row of size s to the top of the diagram (in the case that $r > s + 1$ or $r = s + 1$ and $s < \ell(\lambda)$).

Observe that if the weight of λ is $(-1)^{\ell(\lambda)}$ then since $\phi(\lambda)$ has the same number of cells and either one more or one less row than λ then the weight of $\phi(\lambda)$ is the negative of the weight of λ .

Also observe for each of the 4 cases we have considered, $\phi(\phi(\lambda))$ is just λ . This implies we can say that in the expansion of the expression $\sum_{\lambda \text{ strict}} (-1)^{\ell(\lambda)} q^{|\lambda|}$, the term corresponding to the partition λ will cancel with the term corresponding to the partition $\phi(\lambda)$ because the then $\phi(\lambda)$ will also cancel with $\phi(\phi(\lambda)) = \lambda$.

There are two cases that we have not considered. These terms do not cancel. One is that $r = s$ and $s = \ell(\lambda)$ and so we have a partition of the form $(2m-1, 2m-2, \dots, m)$ and the other is that $r = s+1$ and $s = \ell(\lambda)$ and this is a partition of the form $(2m, 2m-1, \dots, m+1)$. \square

We encourage the reader to take a pencil and draw an arrow between the diagrams of the strict partitions given in the example above to show that the involution works as expected.

REFERENCES

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