PROOF: Note that

$$n = \underbrace{1 + 1 + 1 + \ldots + 1}_{n \text{ terms}} \leq 1 + m + m^2 + \ldots + m^{n-1} = \frac{m^n - 1}{m - 1}$$

$$\leq m^n - 1 < m^n. \qquad \blacksquare$$

EXERCISES

1. Prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Prove that

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2.$$

[Hint: Use Theorem 1-1.]

3. Prove that

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}).$$

4. Prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

5. Prove that

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

6. Prove that

$$\frac{1}{2 \cdot 1} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

7. Suppose that $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and in general $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. (F_n is called the nth Fibonacci number.) Prove that

$$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1.$$

In Exercises 8 through 16, F_n stands for the nth Fibonacci number. (See Exercise 7.)

8. Prove that

$$F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}.$$

9. Prove that

$$F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1.$$

10. Prove that

$$F_{n+1}^2 - F_n F_{n+2} = (-1)^n.$$

11. Prove that

$$F_1F_2 + F_2F_3 + F_3F_4 + \ldots + F_{2n-1}F_{2n} = F_{2n}^2.$$

12. Prove that

$$F_1F_2 + F_2F_3 + F_3F_4 + \ldots + F_{2n}F_{2n+1} = F_{2n+1}^2 - 1.$$

13. The Lucas numbers L_n are defined by the equations $L_1 = 1$, and $L_n = F_{n+1} + F_{n-1}$ for each $n \geq 2$. Prove that

$$L_n = L_{n-1} + L_{n-2} \ (n \geq 3).$$

14. What is wrong with the following argument?

"Assuming $L_n = F_n$ for $n = 1, 2, \ldots, k$, we see that

$$\begin{aligned} L_{k+1} &= L_k + L_{k-1} && \text{(by Exercise 13)} \\ &= F_k + F_{k-1} && \text{(by our assumption)} \\ &= F_{k+1} && \text{(by definition of } F_{k+1}\text{).} \end{aligned}$$

Hence, by the principle of mathematical induction, $F_n = L_n$ for each positive integer n."

15. Prove that $F_{2n} = F_n L_n$.

16. Prove that

$$L_1 + 2L_2 + 4L_3 + 8L_4 + \ldots + 2^{n-1}L_n = 2^n F_{n+1} - 1.$$

17. Prove that $n(n^2-1)(3n+2)$ is divisible by 24 for each positive integer n.

18. Prove that if n is an odd positive integer, then $x+y$ is a factor of x^n+y^n. (For example, if $n=3$, then $x^3+y^3=(x+y)(x^2-xy+y^2)$.)

1–2 THE BASIS REPRESENTATION THEOREM

Early in grade school, you learned to express the integers in terms of the decimal system of notation. The number ten is said to be the *base* for decimal notation, because the digits in any integer are coefficients of the progressive powers of 10.

Example 1–1: In the decimal system, two hundred nine is written 209, which stands for

$$2\cdot 10^2+0\cdot 10^1+9\cdot 10^0.$$

Similarly, for four thousand one hundred twenty-nine we write 4129, which stands for

$$4\cdot 10^3+1\cdot 10^2+2\cdot 10^1+9\cdot 10^0.$$

We can likewise express integers in binary, or base two, notation. In this case the digits 0 and 1 are used as the coefficients of the progressive powers of 2.

Example 1–2: In binary notation, we write twenty-three as 10111, which stands for

$$1\cdot 2^4+0\cdot 2^3+1\cdot 2^2+1\cdot 2^1+1\cdot 2^0,$$

and thirty-six has the form 100100, which stands for

$$1\cdot 2^5+0\cdot 2^4+0\cdot 2^3+1\cdot 2^2+0\cdot 2^1+0\cdot 2^0.$$

The basis representation theorem states that each integer greater than 1 can serve as a base for representing the positive integers.

THEOREM 1–3 *(Basis Representation Theorem): Let k be any integer larger than 1. Then, for each positive integer n, there exists a representation*

$$n=a_0k^s+a_1k^{s-1}+\ldots+a_s, \tag{1-2-1}$$

where $a_0\neq 0$, and where each a_i is nonnegative and less than k. Furthermore, this representation of n is unique; it is called the representation of n to the base k.

REMARK: For each base k, we can also represent 0 by letting all the a_i be equal to 0.

PROOF: Let $b_k(n)$ denote the number of representations of n to the base k. We must show that $b_k(n)$ always equals 1.

It is possible that some of the coefficients a_i in a particular representation of n are equal to zero. Without affecting the representation, we may exclude terms that are zero. Thus suppose that

$$n=a_0k^s+a_1k^{s-1}+\ldots+a_{s-t}k^t,$$

where now neither a_0 nor a_{s-t} equals zero. Then

$$\begin{aligned} n-1&=a_0k^s+a_1k^{s-1}+\ldots+a_{s-t}k^t-1\\ &=a_0k^s+a_1k^{s-1}+\ldots+(a_{s-t}-1)k^t+k^t-1\\ &=a_0k^s+a_1k^{s-1}+\ldots+(a_{s-t}-1)k^t+\sum_{j=0}^{t-1}(k-1)k^j \end{aligned}$$

by Theorem 1–2 with $x=k$. Thus we see that for each representation of n to the base k, we can find a representation of $n-1$. If n has another representation to the base k, the same procedure will yield a new representation of $n-1$. Consequently

$$b_k(n)\le b_k(n-1). \tag{1-2-2}$$

It is important to note that inequality (1–2–2) holds even if n has *no* representation because $b_k(n)=0\le b_k(n-1)$ in that case. Inequality (1–2–2) implies the following inequalities:

$$b_k(n+2)\le b_k(n+1)\le b_k(n),$$
$$b_k(n+3)\le b_k(n+2)\le b_k(n+1)\le b_k(n),$$

and, in general, if $m\ge n+4$,

$$b_k(m)\le b_k(m-1)\le b_k(m-2)\le\ldots\le b_k(n+1)\le b_k(n).$$

Since $k^n>n$ by Corollary 1–1, and since k^n clearly has at least one representation (namely, itself), we see that

$$1\le b_k(k^n)\le b_k(n)\le b_k(1)=1.$$

The extreme entries in this set of inequalities are ones, so that all of the intermediate entries must be equal to 1. Thus $b_k(n) = 1$, and Theorem 1–3 is established. ■

Once a base $k (k > 1)$ has been chosen, we can represent any positive integer n uniquely as a sum of multiples of powers of k:

$$n = a_s k^s + a_{s-1} k^{s-1} + \dots + a_1 k + a_0$$

where $a_0, a_1, \dots, a_s$ stand for nonnegative integers less than k. This representation is usually denoted by the symbol "$a_s a_{s-1} \dots a_1 a_0$" (not a product). For bases less than or equal to ten, the a_i are chosen from the symbols 0, 1, 2, ..., 9 with their usual meanings; however, if k is greater than ten, one must invent additional symbols in order to have a total of k different symbols (one for each of the integers from zero to $k - 1$).

Example 1–3: Let A stand for ten; B, for eleven; C, for twelve; D, for thirteen; E, for fourteen; and F, for fifteen. Using these symbols, we write three hundred to the base sixteen as $12C$, that is,

$$1 \cdot 16^2 + 2 \cdot 16^1 + 12 \cdot 16^0.$$

Similarly, two hundred is represented as $C8$; one hundred, as 64; and ten, as A.

This ability to represent integers to any base greater than one is much more than a mathematical curiosity. The bases 2, 8, and 16 are important in computer science. More useful to us, however, is the applicability of Theorem 1–3 in the proofs of many results about integers. We shall obtain some of these results in the next chapter.

EXERCISES

1. Write the numbers twenty-five, thirty-two, and fifty-six to the base five.
2. Write the numbers forty-seven, sixty-eight, and one hundred twenty-seven to the base 2.
3. What is the least number of weights required to weigh any integral number of pounds up to 63 pounds if one is allowed to put weights in only one pan of a balance?
4. Prove that each nonzero integer may be uniquely represented in the form

$$n = \sum_{j=0}^{s} c_j 3^j,$$

where $c_s \neq 0$, and each c_j is equal to -1, 0, or 1.

5. Using Exercise 4, determine the least number of weights required to weigh any integral number of pounds up to 80 pounds if one is allowed to put weights in both pans of a balance.
6. Prove that if

$$a_s k^s + a_{s-1} k^{s-1} + \dots + a_0$$

is a representation of n to the base k, then $0 < n \leq k^{s+1} - 1$. [Hint: Use Theorem 1–2.]

7. Without using Theorem 1–3, prove directly that two different representations to the base k represent different integers. [Hint: Use Exercise 6.]

Question: Why is it clear that $b_k(1) = 1$?

Outline of theorem 1·3:

Let $b_k(n)$ = # of representations of n by k

A. To show: $b_k(n) = 1$

I. $b_k(n) \geq 1$ Why:

a. $b_k(n) \leq b_k(n-1)$

b. $k^n \geq n$ see corollary 1·1 p.5-6

c. $b_k(k^n) \leq b_k(k^n - 1) \leq \dots \leq b_k(n)$

d. $b_k(k^n) \geq 1$

II. $b_k(n) \leq 1$

a. $b_k(n) \leq b_k(n-1) \leq \dots \leq b_k(1)$

b. Clearly $b_k(1) = 1$ see Question ab