Fundamentals of Mathematics

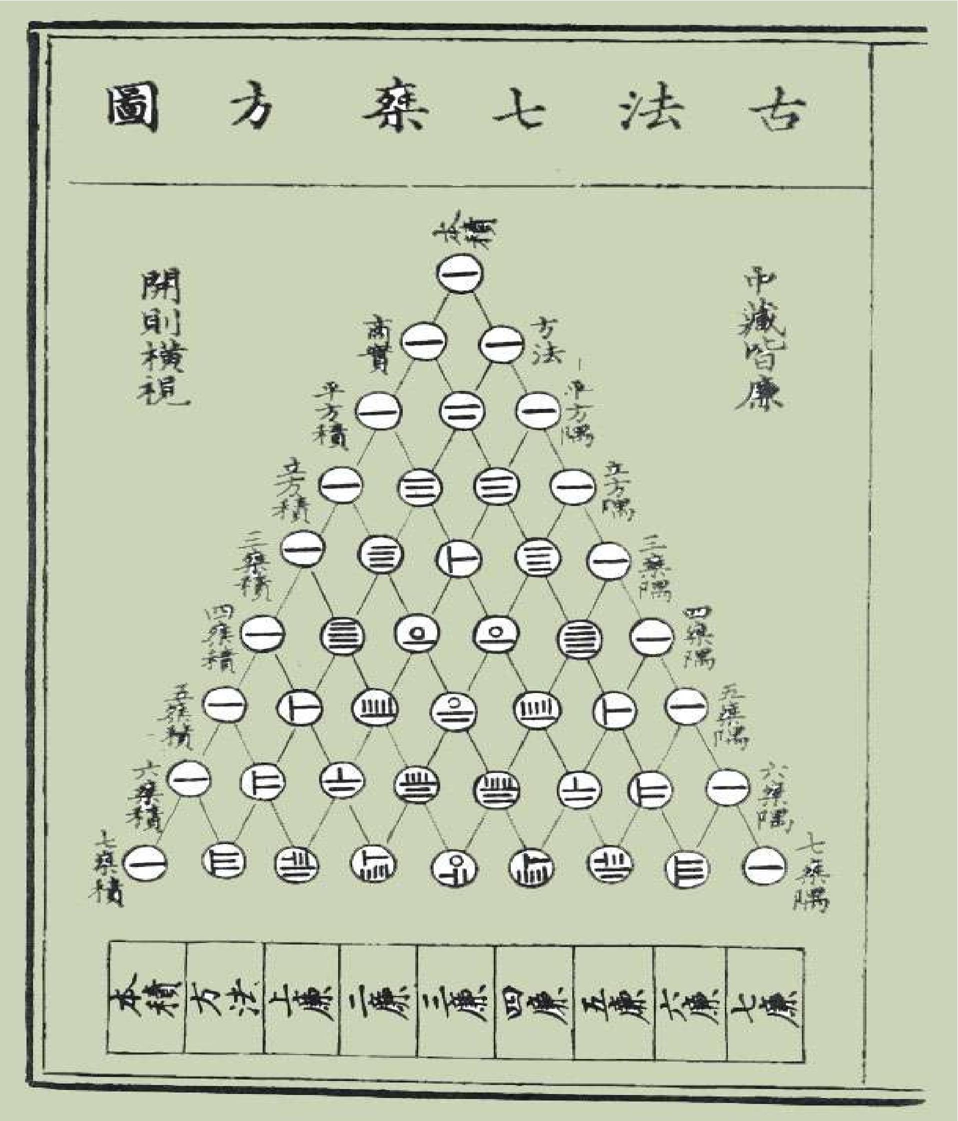
Pascal's Triangle – An Investigation

March 20, 2008 – Mario Soster

Historical Timeline

- A triangle showing the binomial coefficients appear in an Indian book in the 10th century
- In the 13th century Chinese mathematician Yang Hui presents the arithmetic triangle
- In the 16th century Italian mathematician Niccolo Tartaglia presents the arithmetic triangle

Yang Hui's Triangle



Historical Timeline cont…

- Blasé Pascal 1623-1662, a French Mathematician who published his first paper on conics at age 16, wrote a treatise on the 'arithmetical triangle' which was named after him in the 18th century (still known as Yang Hui's triangle in China)
- Known as a geometric arrangement that displays the binomial coefficients in a triangle

Pascal's Triangle

What is the pattern?

What is the next row going to be?

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

We are taking the sum of the two numbers directly above it.

How does this relate to combinations?

- Using your calculator find the value of:

$\binom{5}{0}$	$\binom{5}{1}$	$\binom{5}{2}$	$\binom{5}{3}$	$\binom{5}{4}$	$\binom{5}{5}$
1	5	10	10	5	1

- What pattern do we notice?

It follow's Pascal's Triangle

So, Pascal's Triangle is:

r = 0, r = 1, r = 2, r = 3

n = 0: $\binom{0}{0}$

n = 1: $\binom{1}{0}$ $\binom{1}{1}$

n = 2: $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

n = 3: $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

Pascal's Identity/Rule

- "The sum of the previous two terms in the row above will give us the term below."

$$\begin{matrix} n & & n & & n \quad 1 \\ r & & r \quad 1 & & r \quad 1 \end{matrix}$$

Example 1:

b) How do you simplify $\begin{matrix}11\\4\end{matrix}$ $\begin{matrix}11\\5\end{matrix}$ into a single expression?

b) How do you write $\begin{matrix}12\\3\end{matrix}$ as an expanded expression?

a) Use Pascal’s Identity:

$n \qquad n \qquad n \quad 1$

$r \qquad r \quad 1 \qquad r \quad 1$

$11 \qquad 11$

$4 \qquad 5$

n = 11, and r = 4

$11 \quad 1$

$4 \quad 1$

12

5

b) Use Pascal's Identity:

$n \quad\quad n \quad\quad n \;\; 1$

$r \quad\quad r \;\; 1 \quad\quad r \;\; 1$

12 n + 1 = 12, and r + 1 = 3,

3 so n = 11 and r = 2

11 11

2 3

Or, what is 12 – 3? If you said 9 … try in your calculator:

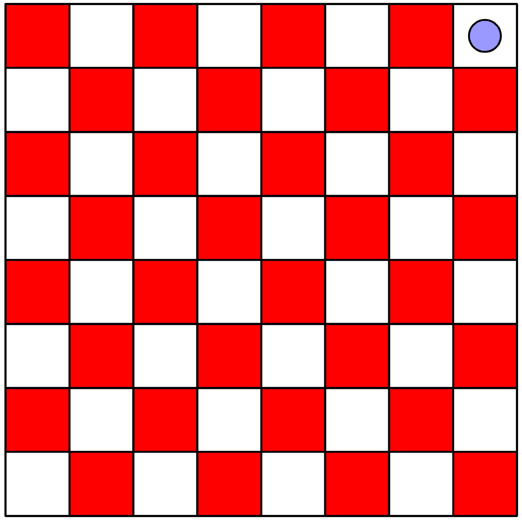
12 12

3 9 They are the same thing!

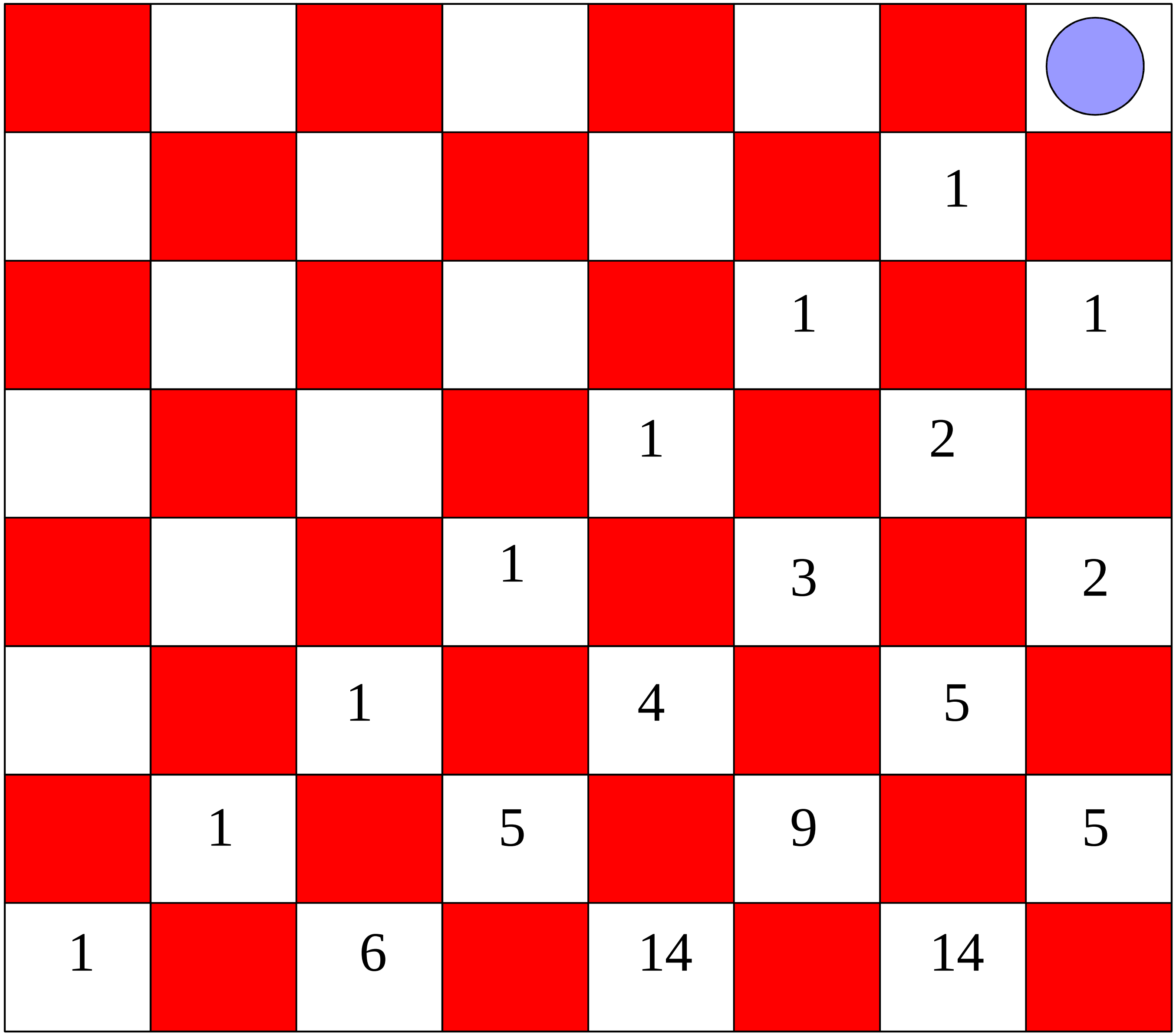
Therefore C(n,r) is equivalent to C(n,n-r)

Example 2:

A former math student likes to play checkers a lot. How many ways can the piece shown move down to the bottom?



Use Pascal's Triangle:



Example 3:

How many different paths can be followed to spell the word 'Fundamentals'?

```
            F
          U   U
        N   N   N
      D   D   D   D
    A   A   A   A   A
  M   M   M   M   M   M
E   E   E   E   E   E   E
  N   N   N   N   N   N
    T   T   T   T   T
      A   A   A   A
        L   L   L
          S   S
```

Use Pascal's Triangle:

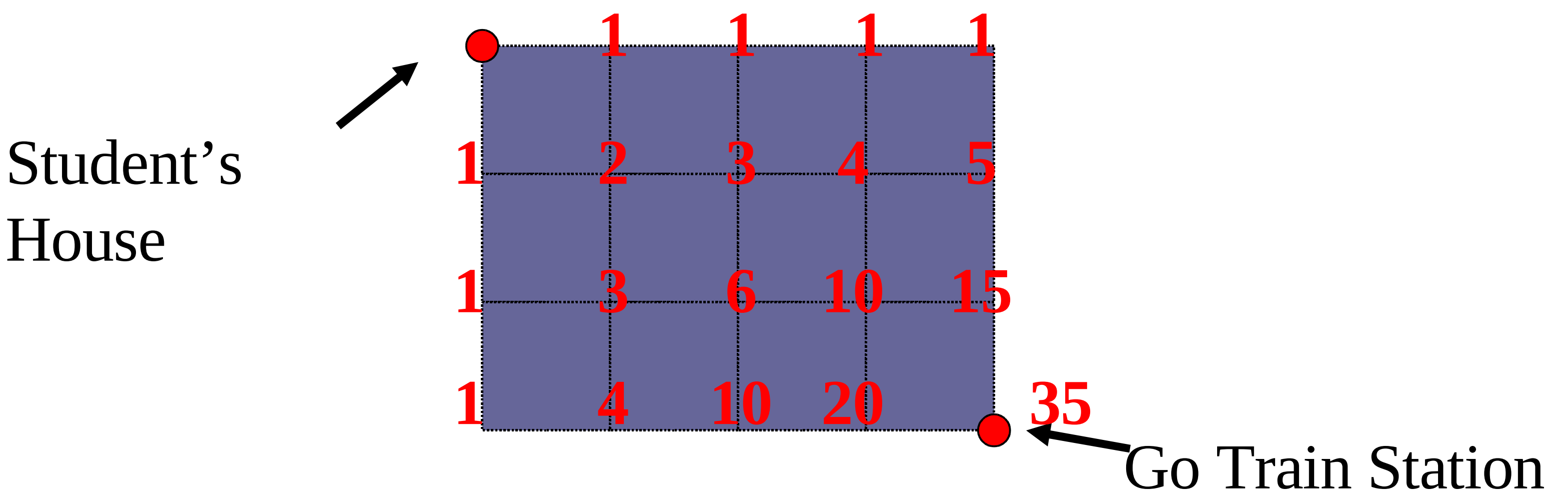
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
7 21 35 35 21 7
28 56 70 56 28
84 126 126 84
210 252 210
462 462

Therefore there are (462 + 462) = 924 total ways.
Using combinations, since there are 12 rows and the final value is in a central position then there C(12,6) = 924 total ways.

Example 3:

The GO Train Station is 3 blocks south and 4 blocks east of a student's house. How many different ways can the student get to the Go Train Station? The student can only go south or east.

Draw a map:



Therefore there are 35 different ways of going from the student's house to the GO Train station.

Note: Using combinations:

C((# of rows + # of columns), (# of rows))

C(7,4) = 35

Try This:

- Expand $(a+b)^4$

$$a^4 \quad 4a^3b \quad 6a^2b^2 \quad 4ab^3 \quad b^4$$

Binomial Theorem

- The coefficients of this expansion results in Pascal's Triangle

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}b^n$$

- The coefficients of the form $\binom{n}{r}$ are called binomial coefficients

Example 4:

Expand $(a+b)^4$

Use the Binomial Theorem:

$$(a \quad b)^4 \quad \genfrac{}{}{0pt}{}{4}{0} a^4 \quad \genfrac{}{}{0pt}{}{4}{1} a^3 b \quad \genfrac{}{}{0pt}{}{4}{2} a^2 b^2 \quad \genfrac{}{}{0pt}{}{4}{3} ab^3 \quad \genfrac{}{}{0pt}{}{4}{4} b^4$$

$$a^4 \quad 4a^3 b \quad 6a^2 b^2 \quad 4ab^3 \quad b^4$$

What patterns do we notice?

- Sum of the exponents in each section will always equal the degree of the original binomial
- The “r” value in the combination is the same as the exponent for the “b” term.

Example 5:

Expand $(2x - 1)^4$

Use the Binomial Theorem:

$$(2x \quad 1)^4 \quad \binom{4}{0}(2x)^4 \quad \binom{4}{1}(2x)^3(\quad 1) \quad \binom{4}{2}(2x)^2(\quad 1)^2 \quad \binom{4}{3}(2x)(\quad 1)^3 \quad \binom{4}{4}(\quad 1)^4$$

$$2^4 x^4 \quad (4)(2^3 x^3)(\quad 1) \quad 6\,(2^2 x^2)(\quad 1)^2 \quad 4\,(2x)(\quad 1)^3 \quad (\quad 1)^4$$

$$16x^4 \quad (4)(8x^3)(\quad 1) \quad (6)(4x^2)(1) \quad (4)(2x)(\quad 1) \quad (1)$$

$$16x^4 \quad 32x^3 \quad 24x^2 \quad 8x \quad 1$$

Example 6:

Express the following in the form $(x+y)^n$

$$\binom{5}{0}a^5 \quad \binom{5}{1}a^4b \quad \binom{5}{2}a^3b^2 \quad \binom{5}{3}a^2b^3 \quad \binom{5}{4}ab^4 \quad \binom{5}{5}b^5$$

Check to see if the expression is a binomial expansion:

$$\binom{5}{0}a^5 \quad \binom{5}{1}a^4b \quad \binom{5}{2}a^3b^2 \quad \binom{5}{3}a^2b^3 \quad \binom{5}{4}ab^4 \quad \binom{5}{5}b^5$$

- The sum of the exponents for each term is constant
- The exponent of the first variable is decreasing as the exponent of the second variable is increasing

n = 5

So the simplified expression is: $(a + b)^5$

General Term of a Binomial Expansion

- The general term in the expansion of (a+b) ^{n} is:

$$t_{r\quad 1} \qquad \begin{matrix} n \\ r \end{matrix} \; a^{n\quad r} b^{r}$$

where r =0, 1, 2, … n

Example 7:

What is the 5th term of the binomial expansion of $(a+b)^{12}$?

Apply the general term formula!

n = 12 ← $(a+b)^{12}$

r = 4 ← 5^{th} term wanted (r +1) = 5

$$t_{r\ \ 1}\quad \begin{matrix} n \\ r \end{matrix}\ a^{n\ \ r} b^{r}$$

$$t_{4\ \ 1}\quad \begin{matrix} 12 \\ 4 \end{matrix}\ a^{12\ \ 4} b^{4}$$

$$t_{5}\quad 495\ a^{8} b^{4}$$

Other Patterns or uses:

- Fibonacci Numbers (found using the ‘shadow diagonals’)
- Figurate Numbers
- Mersenne Number
- Lucas Numbers
- Catalan Numbers
- Bernoulli Numbers
- Triangular Numbers
- Tetrahedral Numbers
- Pentatope Numbers

Sources:

- Grade 12 Data Management Textbooks
- http://en.wikipedia.org/wiki/Pascal%27s_triangle
- http://www.math.wichita.edu/history/topics/notheory.html#pascal
- http://mathforum.org/workshops/usi/pascal/pascal.links.html
- http://mathworld.wolfram.com/PascalsTriangle.html
- http://milan.milanovic.org/math/

(check out this website, select English) or use http://milan.milanovic.org/math/english/contents.html