Squares of Integers

My method first:

The recurrence relation: $a_n - a_{n-1} = n^2 - (n-1)^2 = 2n-1$

$$f(q) = a_0 + a_1q + a_2q^2 + a_3q^3 + \ldots + a_nq^n + \ldots$$
$$q\, f(q) = \quad a_0q + a_1q^2 + a_2q^3 + \ldots + a_{n-1}q^n + \ldots$$

$$f(q) - qf(q) = a_0 + (a_1-a_0)q + (a_2-a_1)q^2 + (a_3-a_2)q^3 + \ldots + (a_n-a_{n-1})q^n + \ldots$$
$$f(q)(1-q) = 0 + q + 3q^2 + 5q^3 + \ldots + (2n-1)q^n + \ldots$$

Now, the recurrence relation: $b_n - b_{n-1} = 2$

$$f(q)(1-q) = q + 3q^2 + 5q^3 + 7q^4 + \ldots + (2n-1)q^n + \ldots$$
$$q\, f(q)(1-q) = \quad q^2 + 3q^3 + 5q^4 + \ldots + (2n-3)q^n + \ldots$$

$$f(q)(1-q) - qf(q)(1-q) = q + 2q^2 + 2q^3 + 2q^4 + \ldots + 2q^n + \ldots$$
$$f(q)(1-q)^2 = q + 2q^2 + 2q^3 + 2q^4 + \ldots + 2q^n + \ldots$$

Now, the recurrence relation: $c_n - c_{n-1} = 0$

$$f(q)(1-q)^2 = q + 2q^2 + 2q^3 + 2q^4 + \ldots + 2q^n + \ldots$$
$$q\, f(q)(1-q)^2 = \quad q^2 + 2q^3 + 2q^4 + \ldots + 2q^n + \ldots$$

$$f(q)(1-q)^2 - q\, f(q)(1-q)^2 = q + q^2$$
$$f(q)(1-q)^3 = q + q^2$$
$$f(q) = (q + q^2)/ (1-q)^3$$

Mike's method (I think):

I can't figure any way to start it except by copying the first part of my solution, so:

$$f(q) = a_0 + a_1q + a_2q^2 + a_3q^3 + \ldots + a_nq^n + \ldots$$
$$q\, f(q) = \quad a_0q + a_1q^2 + a_2q^3 + \ldots + a_{n-1}q^n + \ldots$$

$$f(q) - qf(q) = a_0 + (a_1-a_0)q + (a_2-a_1)q^2 + (a_3-a_2)q^3 + \ldots + (a_n-a_{n-1})q^n + \ldots$$
$$f(q)(1-q) = 0 + q + 3q^2 + 5q^3 + \ldots + (2n-1)q^n + \ldots$$

Then,

$f(q) = \sum n^2q^n = (1/(1-q))*\sum(2n-1)q^n = (1/(1-q))*(2\sum nq^n - \sum q^n)$

$= (1/(1-q))*[2(1/(1-q)^2 - (1/(1-q)) - (1/(1-q)) + 1]$

(Note: the +1 is needed since his sequence $1/(1-q)$ starts with $1 + q + q^2 + \ldots$ which is a subtracted term in my equation. However, my sequence starts with $q + q^2 + \ldots$ so I need to +1 to compensate for the extra -1 from his formula.)

$(1/(1-q))*[2(1/(1-q)^2 - 1/(1-q)) - (1/(1-q)) + 1]$

$= (1/(1-q))*[2q/(1-q)^2 - (1/(1-q)) + 1]$

$= (1/(1-q))*[(2q-(1-q)+(1-q)^2)/(1-q)^2]$

$= (1/(1-q))*[(2q-1+q+1-2q+q^2)/(1-q)^2]$
$= (q + q^2)/(1-q)^3$

Fibonacci – a much easier way

$a_n = a_{n-1} + a_{n-2}$ so: $a_n - a_{n-1} - a_{n-2} = 0$

$$f(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 8q^5 + \ldots + a_n q^n + \ldots$$
$$q*f(q) = q + q^2 + 2q^3 + 3q^4 + 5q^5 + \ldots + a_{n-1} q^n + \ldots$$
$$q^2*f(q) = q^2 + q^3 + 2q^4 + 3q^5 + \ldots + a_{n-2} q^n + \ldots$$

$f(q) - q*f(q) - q^2*f(q) = 1$
$f(q) (1 – q – q^2) = 1$
$f(q) = 1/(1 – q – q^2)$