SOME FIBBONACCI GENERATING FUNCTIONS

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Recall $F(q) = \sum_{n\geq 0} F_{n+1}q^n = \frac{1}{1-q-q^2}$ and $L(q) = \sum_{n\geq 0} L_{n+1}q^n = \frac{1+2q}{1-q-q^2}$

(1) Use the fact that $(A(q) + A(-q))/2 = \sum_{n\geq 0} a_{2n}q^{2n}$ to give a generating function for the odd Fibbonacci numbers $F_{odd}(q) = \sum_{n\geq 0} F_{2n+1}q^n$.
(2) Use the fact that $(A(q) - A(-q))/2 = \sum_{n\geq 0} a_{2n+1}q^{2n+1}$ to give a generating function for the even Fibbonacci numbers $F_{even}(q) = \sum_{n\geq 0} F_{2n+2}q^n$.
(3) Use the following set of three formulas:

$$F_n^2 = F_nF_{n-1} + F_nF_{n-2}$$
$$F_nF_{n+1} = F_n^2 + F_nF_{n-1}$$
$$F_{n+2}F_n = F_{n+1}F_n + F_n^2$$

to write down three equations with the generating functions $D^{(0)}(q) = \sum_{n\geq 0} F_{n+1}^2 q^n$, $D^{(1)}(q) = \sum_{n\geq 0} F_{n+1}F_{n+2}q^n$, $D^{(2)}(q) = \sum_{n\geq 0} F_{n+1}F_{n+3}q^n$. Use those equations to solve for $D^{(0)}(q)$, $D^{(1)}(q)$, $D^{(2)}(q)$.
(4) Use the results of the previous problem and the fact that $L_n = F_{n+1} + F_{n-1}$ for $n \geq 2$ to give a formula for the generating function $M^{(0)}(q) = \sum_{n\geq 0} F_{n+1}L_{n+1}q^n$.
(5) Use the generating functions $D^{(0)}(q)$, $D^{(1)}(q)$, $D^{(2)}(q)$ to give a formula for the generating function $M^{(1)}(q) = \sum_{n\geq 0} F_{n+2}L_{n+1}q^n$.
(6) Use the previous two problems and the fact that $L_{n+2} = L_{n+1} + L_n$ to find a formula for the generating function $M^{(-1)}(q) = \sum_{n\geq 0} F_{n+1}L_{n+2}q^n$

Record your answers below:

$$F_{odd}(q) = 1 + 2q + 5q^2 + 13q^3 + 34q^4 + \cdots = \sum_{n\geq 0} F_{2n+1}q^n =$$

$$F_{even}(q) = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n\geq 0} F_{2n+2}q^n =$$

$$D^{(0)}(q) = 1 + q + 4q^2 + 9q^3 + 25q^4 + \cdots = \sum_{n\geq 0} F_{n+1}^2 q^n =$$

$$D^{(1)}(q) = 1 + 2q + 6q^2 + 15q^3 + 40q^4 + \cdots = \sum_{n\geq 0} F_{n+2}F_{n+1}q^n =$$

$$D^{(2)}(q) = 2 + 3q + 10q^2 + 24q^3 + 65q^4 + \cdots = \sum_{n\geq 0} F_{n+3}F_{n+1}q^n =$$

$$M^{(0)}(q) = 1 + 3q + 8q^2 + 21q^3 + 55q^4 + \cdots = \sum_{n\geq 0} F_{n+1}L_{n+1}q^n =$$

$$M^{(1)}(q) = 1 + 6q + 12q^2 + 35q^3 + 88q^4 + \cdots = \sum_{n\geq 0} F_{n+2}L_{n+1}q^n =$$

$$M^{(-1)}(q) = 3 + 4\,q + 14\,q^2 + 33\,q^3 + 90\,q^4 + \cdots = \sum_{n\geq 0} F_{n+1}L_{n+2}q^n =$$