PROOF OF A GENERATING FUNCTION EXPRESSION

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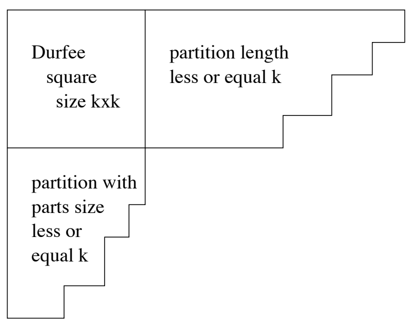
I wanted to give you an example of an generating function identity which we are able to justify in words now just using addition an multiplication principles of generating functions.

Look at the following diagram for a partition divided into pieces.



Note that

$$q^{k^2} \prod_{i=1}^{k} \frac{1}{(1-q^i)^2}$$

is the generating function for triples consisting of a Durfee square of size k, a partition of length less than or equal to k (which we showed in class to have generating function $\prod_{i=1}^{k} \frac{1}{1-q^i}$), and a partition with parts smaller than or equal to k (which we also showed in class to have a generating function $\prod_{i=1}^{k} \frac{1}{1-q^i}$).

Now the generating function for every partition (with generating function we have shown in class to be $\prod_{i\geq 1} \frac{1}{1-q^i}$) is either empty, or it has a Durfee square of size 1, or it has a Durfee square of size 2, or it has a Durfee square of size 3, etc. So by the addition principle of generating functions we have

$$\prod_{i\geq 1} \frac{1}{1-q^i} = 1 + \sum_{k\geq 1} \frac{q^{k^2}}{\prod_{i=1}^{k}(1-q^i)^2} \ .$$

This is an infinite identity but the partial expansions of of the generating function expression should be able to tell us what the first 10 or so terms are. So if we go to a computer and ask for an expansion of the first terms of both of these generating functions they should agree.

```
sage:  q = var('q')
sage:  taylor( mul(1/(1-q^(i+1)) for i in range(10)),q,0,11)
1 + q + 2*q^2 + 3*q^3 + 5*q^4 + 7*q^5 + 11*q^6 + 15*q^7 + 22*q^8
+ 30*q^9 + 42*q^10 + 55*q^11

sage:  taylor( 1+q/(1-q)^2+q^4/(1-q)^2/(1-q^2)^2+q^9/(1-q)^2/(1-q^2)^2/(1-q^3)^2,
q, 0, 11)
1 + q + 2*q^2 + 3*q^3 + 5*q^4 + 7*q^5 + 11*q^6 + 15*q^7 + 22*q^8
+ 30*q^9 + 42*q^10 + 56*q^11
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Notice that these two answers differ in the coefficient of q^{11} this is because the first expression is not correct at the 11^{th} term because it doesn't have $1/(1-q^{11})$.